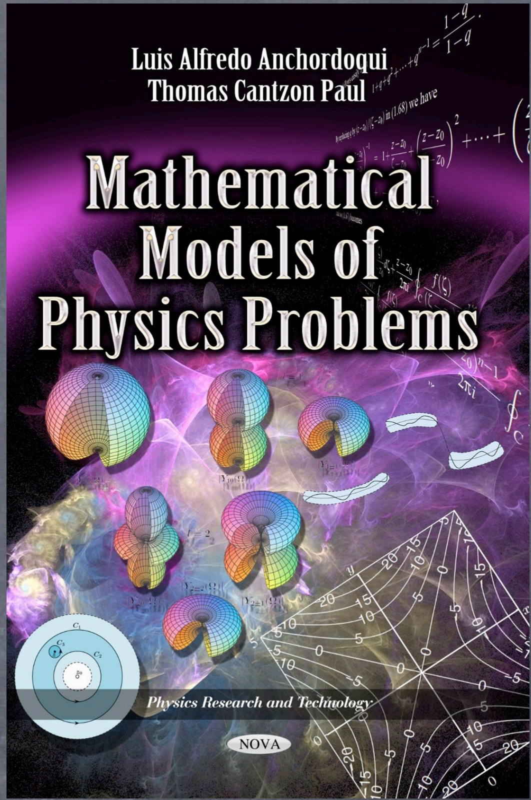
Physics 307

Mathematical Physics

Luis Anchordoqui

Bibliography



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Provisional Course Outline

(Please note this may be revised during the course
to match coverage of material during lectures, etc.)

1st week - Analytic Functions

2rd week - Integration in the Complex Plane

3rd week - Isolated Singularities and Residues

4th week - Elements of Linear Algebra

5th week - Initial Value Problem (Picard's Theorem)

6th week - Initial Value Problem (Green Matrix)

7th week - Boundary Value Problem (Sturm-Liouville Operator)

8th week - Boundary Value Problem (Special Functions)

9th week - Fourier Series and Fourier Transform

10th week - Hyperbolic Partial Differential Equation (Wave equation)

11th week - Parabolic Partial Differential Equation (Diffusion equation)

12th week - Elliptic Partial Differential Equation (Laplace equation)

Midterm-exams (October 6, November 7, December 12)

Final-exam (December 19 -- 3:45 pm to 5:45 pm)

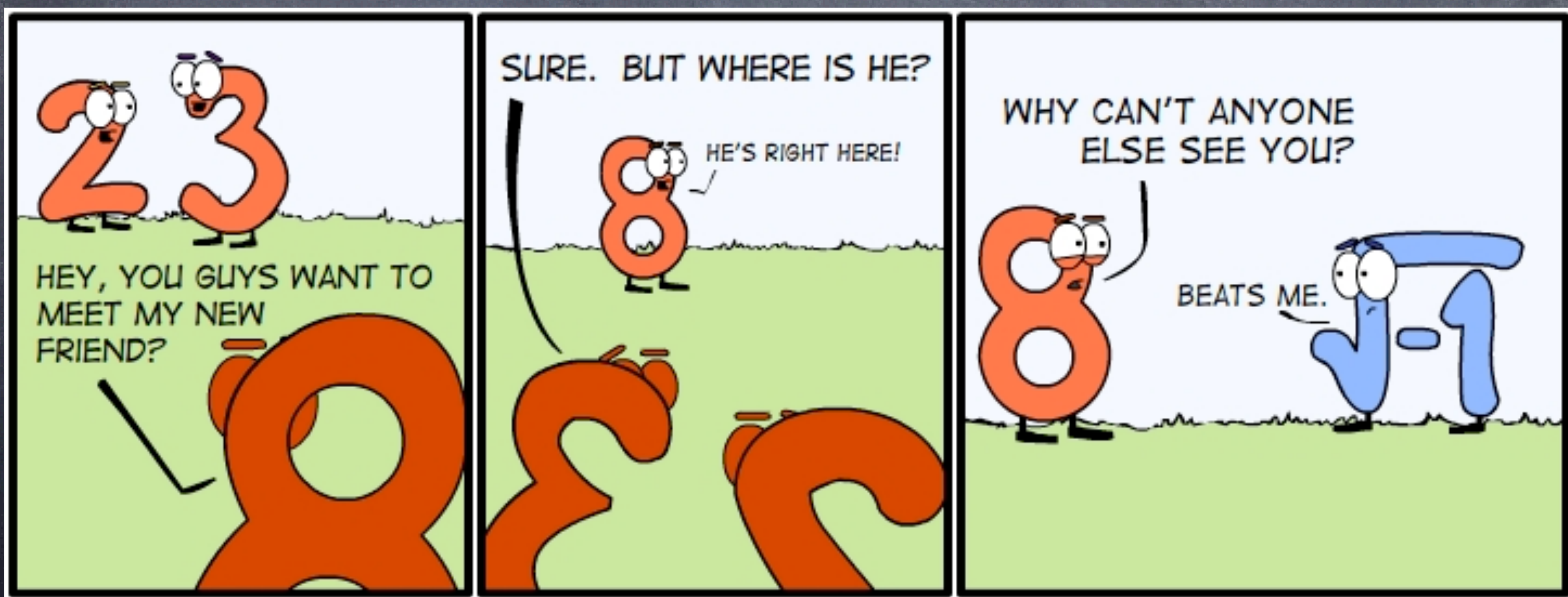
Complex Analysis I

1.1 Complex Algebra

1.2 Functions of a Complex Variable

1.3 Cauchy's Theorem and its Applications

1.4 Isolated Singularities and Residues



Complex Algebra

Real number system is adequate for solving many mathematical and physical problems

It is necessary to extend such a system to solve equation $x^2 + 1 = 0$ because when we square a real number we get a nonnegative number

Definition 1.1.1.

We define i to be imaginary number equal to square root of -1

That is $i = \sqrt{-1}$ ☛ which implies $i^2 = -1$

Proposition 1.1.1.

We can combine the set real numbers $\mathbb{R}$

with this new imaginary number

to form set complex numbers $\mathbb{C}$

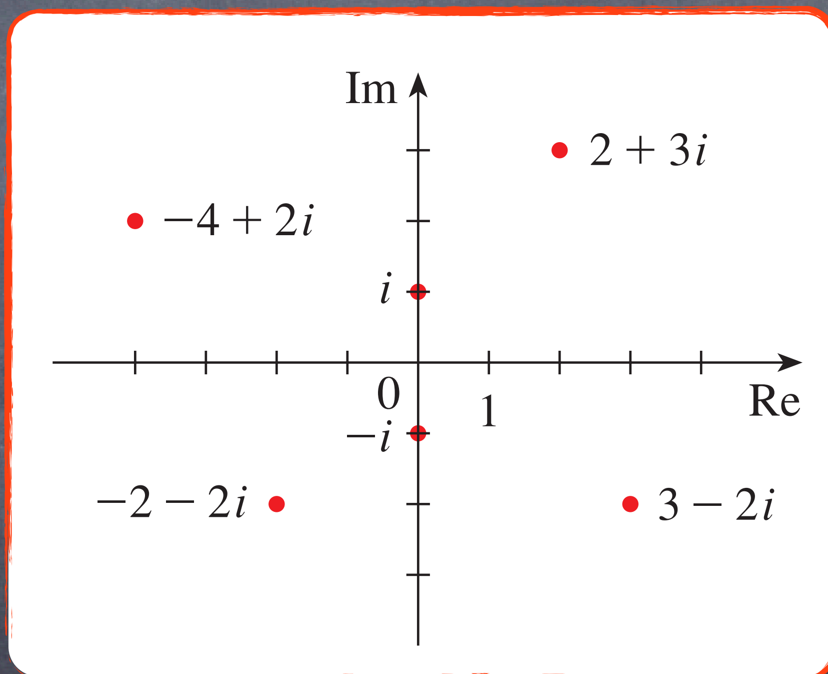
Definition 1.1.2.

A complex number z is an ordered pair (x, y) with $x \in \mathbb{R}, y \in \mathbb{R}$

x is called real part of z ☛ $x = \Re\mathrm{e}$

y is called imaginary part of z ☛ $y = \Im\mathrm{m}\ z$

Geometric representation of z as a point in complex plane



Herein $\mathbb{C}$ denotes set of all complex numbers

$$\mathbb{C} = \{z : z = x + iy, x \in \mathbb{R}, y \in \mathbb{R}\}$$

Proposition 1.1.2.

Addition and subtraction is defined exactly as in $\mathbb{R}^2$

for example ☛ if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$
then we define $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Multiplication makes $\mathbb{C}$ different from $\mathbb{R}^2$

We define

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

We can define division of complex numbers

if $z_2 \neq 0$ then we define

$$\frac{1}{z_2} = \frac{x_2 - iy_2}{x_2^2 + y_2^2}$$

and therefore

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

Definition 1.1.3.

If $z = x + iy$ is a complex number

then its conjugate is defined by $z^* = x - iy$

Remark 1.1.1.

Conjugation has following properties

which follow directly from the definition:

$$\Re\mathrm{e}\, z = (z + z^*)/2$$

$$\Im\mathrm{m}\, z = -(z - z^*)/2$$

$$(z_1 + z_2)^* = z_!^* + z_2^*; (z_1 z_2)^* = z_1^* z_2^*$$

it follows from this last property that if $\lambda \in \mathbb{R}$ ☛ $(\lambda z)^* = \lambda z^*$

Remark 1.1.2.

Unlike real numbers

complex numbers do not have a natural ordering

so there is no analog of complex-valued inequalities

Proposition 1.1.3.

Let $z = x + iy$ be a complex number with x and y both nonzero

exists $r \in (0, \infty)$ and $\vartheta \in (-\pi, \pi]$

such that $z = r\, e^{i\vartheta}$ with $e^{i\vartheta} = \cos\vartheta + i \sin\vartheta$

Coordinates of polar form of z

are related to its Cartesian components according to

$$r = |z| = \sqrt{x^2 + y^2} \quad \text{and} \quad \vartheta = \tan^{-1}(y/x)$$

ϑ ☛ principal argument of z usually written as $\vartheta = \operatorname{Arg} z$

Reason to restrict ϑ in $(-\pi, \pi]$ is to get uniqueness of representation

2π rotation does not change the point $\operatorname{Arg} z = \arg z + 2k\pi$

integer

Theorem 1.1.1. De Moivre's theorem

If $z = r\,(\cos\vartheta + i \sin\vartheta)$ and n is a positive integer

☛ $z^n = [r(\cos\vartheta + i\sin\vartheta)]^n = r^n(\cos n\vartheta + i \sin n\vartheta)$

This says that to take n-th power of a complex number

we take n-th power of the modulus and multiply argument by n

Corollary 1.1.1.

De Moivre's Theorem ☛

n-th root of complex number z is a complex number w such that $w^n = z$

writing these two numbers in polar form

$w = s(\cos\varphi + i\sin\varphi)$ & $z = r(\cos\vartheta + i\sin\vartheta)$

Using De Moivre's ☛ $s^n(\cos n\varphi + i\sin n\varphi) = r(\cos\vartheta + i\sin\vartheta)$

equality of these two complex numbers shows that

$s = r^{1/n}, \cos n\varphi = \cos\vartheta$ and $\sin n\varphi = \sin\vartheta$

sine and cosine have period 2π ☛ $n\varphi = \vartheta + 2k\pi$

complex number z has n distinct roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\vartheta + 2k\pi}{n}\right) + i\sin\left(\frac{\vartheta + 2k\pi}{n}\right)\right] \quad (1.1.1.)$$

with $k = 0, 1, \ldots, n-1$

Remark 1.1.3.

Notice that each of n-th roots of z has modulus $|w_k| = r^{1/n}$

all n-th roots of z lie on circle of radius $r^{1/n}$

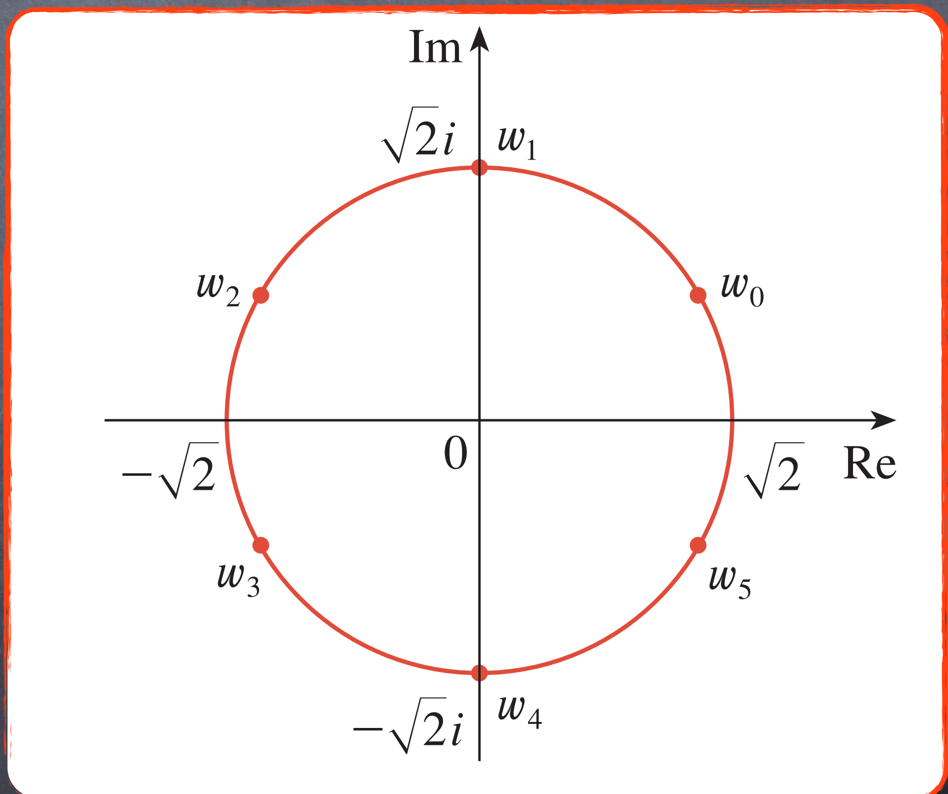
since argument of each successive n-th root exceeds argument of previous root by $2\pi/n$

we see that n-th roots of z are equally spaced on this circle

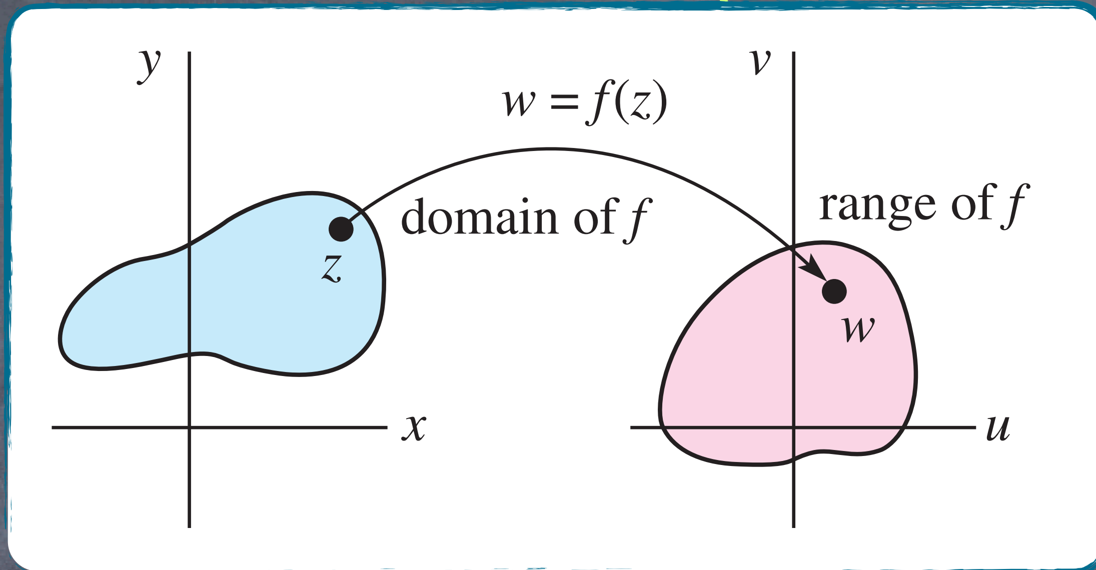
Example 1.1.1.

Six sixth roots of $z = -8$

are shown here ☛



Functions of a Complex Variable



Definition 1.2.1.

A function f from a set A to a set B is a rule of correspondence that assigns to each element in A one and only one element in B

When domain A is a set of complex numbers we say that mapping is a function of a complex variable (or a complex function for short) which we denote $w = f(z) = u(x,y) + iv(x,y)$ (1.2.2.)

Functions u and v can be thought of as real valued functions defined on subsets of $\mathbb{R}^2$ ☛ $u = \Re\text{e}\, f$ and $v = \Im\text{m}\, f$

Proposition 1.2.1.

Given $z_0 \in \mathbb{C}$ and $r > 0$ we denote ball of radius r around z_0

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

Definition 1.2.2.

Let $A \subset \mathbb{C}, B_r(z_0) \subset A$ and $f : A \to \mathbb{C}$

Then f is differentiable at z_0 if limit

$$\lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{z + \delta z - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z_0) \quad (1.2.3.)$$

is independent of direction of approach to point z_0

Recall that for a single real variable we require that

right-hand limit (where one approaches x_0 from $x > x_0$)

and left-hand limit (approaching x_0 from $x < x_0$)

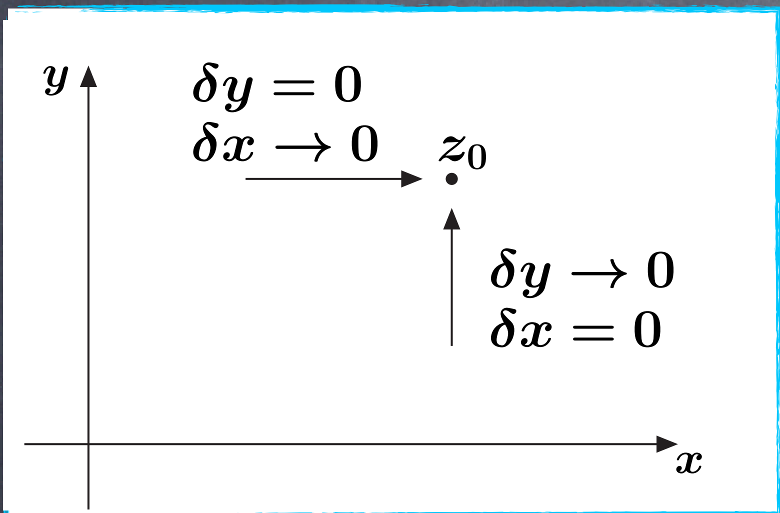
be equal for derivative $df(x)/dx$ to exist at $x = x_0$

For $z = z_0$ ☛ some point in a plane
approach must be generalized

Let δx & δy be increments of x & y ➡ $\delta z = \delta x + i\delta y$

writing $f = u + iv$ ☛ $\delta f = \delta u + i\delta v$ and so $\dfrac{\delta f}{\delta z} = \dfrac{\delta u + i\delta v}{\delta x + i\delta y}$

Take limit (1.2.3.) using two different directions of approach



$\delta y = 0$ ☛ $\delta x \to 0$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$\delta x = 0$ ☛ $\delta y \to 0$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If derivative $\dfrac{df}{dz}$ exists at z_0 ☛ these limits must be identical

Equating real & imaginary parts ☛ Cauchy-Riemann conditions

$$u_x = v_y \qquad u_y = -v_x$$

$$\frac{\partial u}{\partial x} \equiv u_x, \quad \frac{\partial u}{\partial y} \equiv u_y, \quad \frac{\partial v}{\partial x} \equiv v_x, \quad \frac{\partial v}{\partial y} \equiv v_y$$

We have seen that for $f'(z_0)$ to exist ☛ CR conditions must be satisfied

Conversely ☛ if CR conditions hold and partial derivatives of $u(x,y)$ and $v(x,y)$ are continuous

derivative $f'(z) = u_x + iv_x$ exists

This can be seen by writing

$$\delta f = (u_x + iv_x)\delta x + (u_y + iv_y)\delta y \qquad (1.2.11.)$$

dividing by δz

$$\begin{aligned} \frac{\delta f}{\delta z} &= \frac{(u_x + iv_x)\delta x + (u_y + iv_y)\delta y}{\delta x + i\delta y} \\ &= \frac{(u_x + iv_x) + (u_y + iv_y)\delta y/\delta x}{1 + i\delta y/\delta x} \end{aligned} \qquad (1.2.12.)$$

If $\delta f/\delta z$ is to have a unique value

then dependence on $\delta y/\delta x$ has to be eliminated

Applying Cauchy-Riemann conditions to y derivatives yields

$$u_y + iv_y = -v_x + iu_x \qquad (1.2.13.)$$

Substituting (1.2.13.) into (1.2.12.) cancels out $\delta y/\delta x$ dependence

and leaves us with ☛ $$\frac{\delta f}{\delta z} = u_x + iv_x \qquad (1.2.14.)$$

which shows that $\delta f/\delta z$ is independent of direction of approach provided partial derivatives are continuous

Hence ☛ $f'(z)$ exists

Definition 1.2.3.

If $f(z)$ is differentiable at $z = z_0$

and in some small region around z_0

we say that $f(z)$ is analytic or holomorphic at $z = z_0$

In addition ☛ if $f'(z_0) \neq 0$

we say that $f(z)$ is conformal at z_0

If $f(z)$ is analytic everywhere in (finite) complex plane

we call it an entire function

If $f'(z)$ does not exist at $z = z_0$

then z_0 is labeled a singular point

Example 1.2.1.

Let $f(z) = z^2$

Multiplying $(x+iy)(x+iy) = x^2 - y^2 + 2ixy$

we identify real part $u(x,y) = x^2 - y^2$

and imaginary part $v(x,y) = 2xy$

From (1.2.9.)

$$u_x = 2x = v_y \quad \text{and} \quad u_y = -2y = -v_x \qquad (1.2.15.)$$

We see that $f(z) = z^2$ satisfies Cauchy--Riemann conditions

throughout complex plane

Since partial derivatives are evidently continuous

we conclude that $f(z) = z^2$ is analytic

Example 1.2.2.

Let $f(z) = z^*$

Now $u = x$ and $v = -y$

Applying Cauchy--Riemann conditions

$$u_x = 1 \neq v_y = -1$$

Cauchy--Riemann conditions are not satisfied

and $f(z) = z^*$ is not an analytic function of z

However ☛ note that $f(z) = z^*$ is continuous

providing example of a function that is everywhere continuous but nowhere differentiable in complex plane

Definition 1.2.4.

Let $z = x + iy$ $\quad x, y \in \mathbb{R}$

the exponential function e^z is defined for every $z \in \mathbb{C}$

$$e^z = e^x(\cos y + i \sin y) \qquad (1.2.17.)$$

if we write $e^z = u(x, y) + iv(x, y)$

$u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$

e^z is an entire function

Easy to check that Cauchy-Riemann conditions are satisfied for every $z \in \mathbb{C}$

Remark 1.2.3.

Note that

$$\frac{d}{dz}e^z = u_x + iv_x = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^z \qquad (1.2.18.)$$

so that e^z is its own derivative

Corollary 1.2.2.

For every $y_1, y_2 \in \mathbb{R}$ we have

$$\begin{aligned} e^{i(y_1+y_2)} &= \cos(y_1+y_2) + i\sin(y_1+y_2) \\ &= (\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \qquad (1.2.19.) \\ &= e^{iy_1}e^{iy_2} \end{aligned}$$

On the other hand ☛ if $x_1, x_2 \in \mathbb{R}$

$$e^{x_1+x_2}e^{i(y_1+y_2)} = (e^{x_1}e^{x_2})(e^{iy_1}e^{iy_2}) = (e^{x_1}e^{iy_1})(e^{x_2}e^{iy_2}) \qquad (1.2.20.)$$

writing $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

we deduce addition formula $e^{z_1+z_2} = e^{z_1}e^{z_2}$

Remark 1.2.4.

Note that $|e^z| = |e^x(\cos y + i\sin y)| = e^x|\cos y + i\sin y| = e^x$

Since e^x is never zero it follows that

e^z is non-zero for every $z \in \mathbb{C}$ i.e. $\exp z : \mathbb{C} \to \mathbb{C}_0$

Corollary 1.2.3.

Exponential function e^z is periodic in $\mathbb{C}$

$$\omega = 2\pi k i \qquad k \in \mathbb{Z} \quad \text{is non-zero}$$

It follows that

$$|w| = e^x \qquad \text{and} \qquad \arg\, w = y + 2k$$

Usually we make choice $\arg\, w = y$ with restriction that $-\pi < y \leq \pi$

This restriction means that z lies on horizontal strip

$$\mathcal{A}_\star = \{z \in \mathbb{C} : -\infty < x < \infty, -\pi < y \leq \pi\} \qquad (1.2.21.)$$

Restriction $-\pi < \arg\, w \leq \pi$ can also be indicated

on complex w-plane by a cut along negative real axis

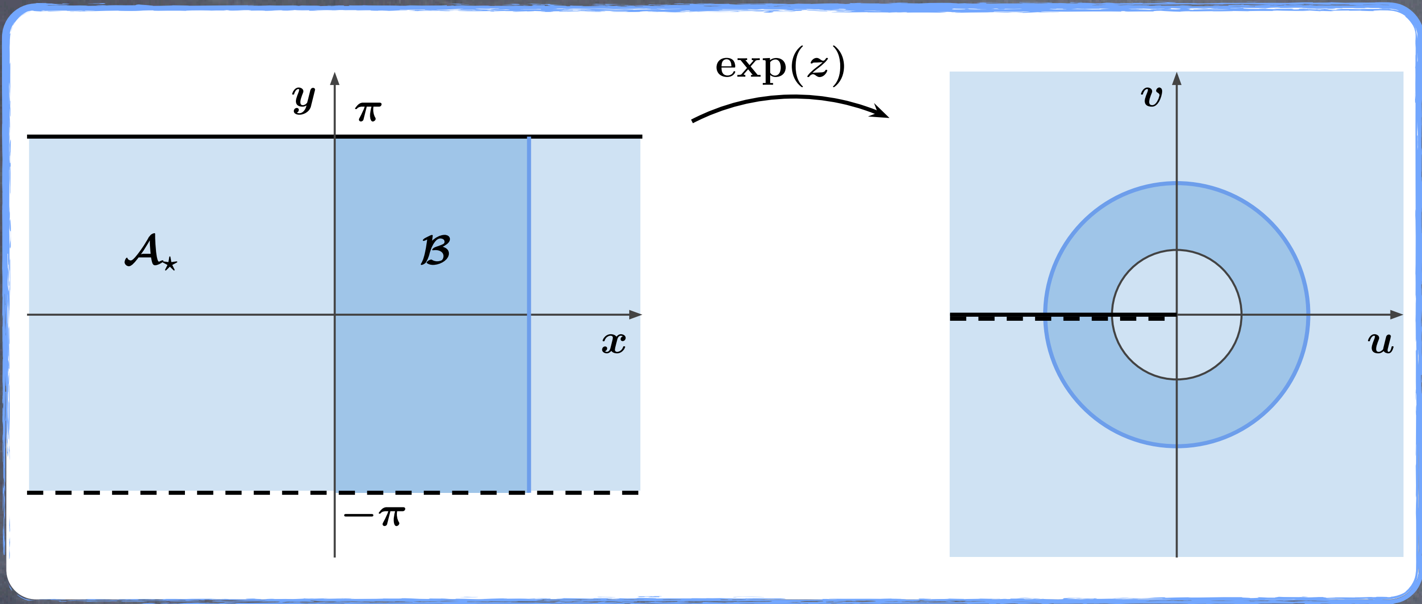
Upper edge of cut ☛ corresponding to $\arg\, w = \pi$

is regarded as part of cut w-plane

Lower edge of cut ☛ corresponding to $\arg\, w = -\pi$

is not regarded as part of cut w-plane

Effect of $f(z) = e^z$ on two domains



Fundamental region of exponential function $A_\star$
is mapped in a one-to-one fashion
on a plane that is cut along negative real axis

Rectangle $\mathcal{B} = \{0 < x < x_0, \; -\pi < y \leq \pi\}$ of height 2π
is mapped in a one-to-one fashion
on an annulus that is cut along negative real axis

Remark 1.2.5.

Region $\mathcal{A}_\star$ is usually known as a fundamental region

Easy to see that every set of type

$$\mathcal{A}_k = \{z \in \mathbb{C} : -\infty < x < \infty, (2k-1)\pi < y \leq (2k+1)\pi\} \qquad k \in \mathbb{Z} \quad (1.2.22.)$$

has same representation than $\mathcal{A}_\star$

Corollary 1.2.4.

Since $\exp : \mathcal{A}_\star \to \mathbb{C}_0$ is one-to-one and onto

there is an inverse function

Definition 1.2.5.

Function $f : \mathbb{C}_0 \to \mathcal{A}_\star$

$$\text{Log } w = z \text{ with } z \in \mathcal{A}_\star \text{ and } e^z = w \qquad (1.2.23.)$$

is called principal logarithmic function

Suppose that $z = x + iy$ and $w = u + iv$ ☛ $x, y, u, v \in \mathbb{R}$

Suppose further that we impose restriction $-\pi < y \leq \pi$

If $w = e^z$ from (1.2.17.) ☛ $u = e^x \cos y$ and $v = e^x \sin y$

$$|w| = (u^2 + v^2)^{1/2} = e^x \quad \text{and} \quad y = \text{Arg } w \qquad (1.2.24.)$$

$\text{Arg } w$ ☛ principal argument of w

Therefore ☛ $x = \ln |w|$ and $y = \text{Arg } w$

or equivalently

$$\text{Log} w = \ln |w| + i \text{ Arg } w \qquad (1.2.25.)$$

In many practical situations one can define $\log w = \ln |w| + i \arg w$ where argument is chosen in order to make logarithmic function continuous in its domain of definition (if this is at all possible)

Definition 1.2.6.

Suppose that $z \in \mathbb{C}$

Then trigonometric functions $\cos z$ and $\sin z$

are defined in terms of exponential the function

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad (1.2.26.)$$

Since exponential function is an entire function

from (1.2.26) both $\cos z$ and $\sin z$ are entire functions

Easy 2C from (1.2.26.) that

$$\frac{d}{dz} \cos z = -\sin z \quad \text{and} \quad \frac{d}{dz} \sin z = \cos z \qquad (1.2.27.)$$

We can define functions $\tan\ z,\ \cot\ z,\ \sec\ z, \csc\ z$

in terms of functions $\cos z$ and $\sin z$ as in real variables

Definition 1.2.7.

Suppose that $z \in \mathbb{C}$

Then hyperbolic functions $\cosh z$ and $\sinh z$ are defined in terms of exponential function

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2} \qquad (1.2.28.)$$

Since exponential function is an entire function from (1.2.28.) both $\cosh z$ and $\sinh z$ are entire functions

Easy 2C from (1.2.28.) that

$$\frac{d}{dz}\cosh z = \sinh z \quad \text{and} \quad \frac{d}{dz}\sinh z = \cosh z \qquad (1.2.29.)$$

We can define functions $\tanh\, z,\ \coth\, z,\ \text{sech}\, z, \text{csch}\, z$ in terms of functions $\cosh z$ and $\sinh z$ as in real variables

Remark 1.2.6. Comparing (1.2.26.) and (1.2.28.) ☛ we obtain

$$\cosh z = \cos iz \quad \text{and} \quad \sinh z = i \sin iz \qquad (1.2.30.)$$

Definition 1.2.8.

A series of form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad ☛ \quad a_n \in \mathbb{C} \text{ and } z_0 \in \mathbb{C}$$

is called a power series around point z_0

Theorem 1.2.2.

If a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \in \mathbb{C}$

then it converges for all $z \in \mathbb{C}$ such that $|z| < |z_0|$

(which is a disc without boundary around origin with radius $|z_0|$)

Proof.

It follows from hypothesis that there exist $M \geq 0$

such that $|a_n z_0^n| \leq M$ for all $n \in \mathbb{N}$

$$|a_n z^n| = |a_n z_0|^n \left|\frac{z}{z_0}\right|^n \leq M \left|\frac{z}{z_0}\right|^n \quad (1.2.31.)$$

Remark 1.2.7.

Note that if series $\displaystyle\sum_{n=0}^{\infty} a_n z_0^n$ diverges then so does series for $|z| > |z_0|$

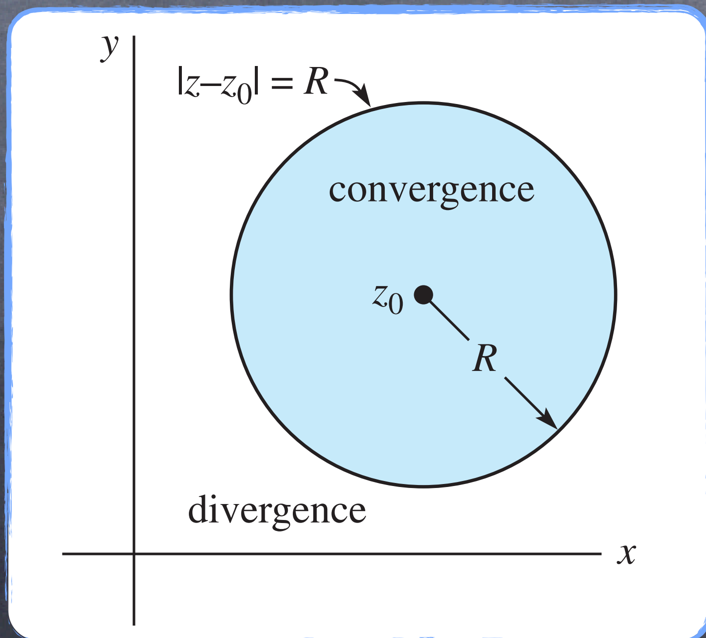
Definition 1.2.9.

Radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z_0^n$

defined as $$R = \max\{|z| : \sum_{n=0}^{\infty} a_n z^n, \text{ converges}\} \quad (1.2.32.)$$

$R \geq 0$ ☛ BUT of course it is possible that $R = 0$

$|z| < R$ (or $|z| > R$) then power series converges (or diverges)



Circle of a convergence of a power series

Theorem 1.2.3.

If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$

then series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has precisely same radius of convergence

Proof.

Fix z_0 with $|z_0| < R$ and pick z such that $|z| < |z_0| < R$

Series $\sum a_n z_0^n$ converges and therefore $\lim_{n\to\infty} a_n z_0^n = 0$

We can thus find a number M such that $|a_n z_0^n| \leq M, \ \forall n$

We now write

$$n a_n z^{n-1} = n a_n z^{n-1} \left(\frac{z_0}{z_0}\right)^n = \frac{n}{z_0} a_n z_0^n \left(\frac{z}{z_0}\right)^{n-1} \qquad (1.2.33.)$$

Introducing $\rho = |z/z_0| < 1$ we have

$$|n a_n z^{n-1}| = |\frac{n}{z_0} a_n z_0^n \rho^{n-1}| \leq \frac{M}{|z_0|} n \rho^{n-1} \qquad (1.2.34.)$$

Now ☛ since

$$\lim_{n\to\infty}\frac{n+1}{n}\rho=\rho<1 \quad (1.2.35.)$$

series $\displaystyle\sum_{n=0}^{\infty} n a_n \rho^{n-1} z_0^{n-1}$ converges

Since z_0 was an arbitrary number with $|z_0|<R$

series converges uniformly for $|z|<R$

This shows that radius of convergence R'

of series of derivatives satisfies $R'\geq R$

Theorem 1.2.4.

If $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R>0$

then ☛ $\displaystyle F(z)=\sum_{n=0}^{\infty} a_n z^n$ is differentiable on $S=\{z\in\mathbb{C}:|z|<R\}$

and derivative is $\displaystyle f(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$

Proof.

We will show that

$$|[F(z+h)-F(z)]/h - f(z)| \to 0 \text{ as } h \to 0 \quad \text{whenever} \quad |z| < R$$

Using binomial theorem ☛ $(z+h)^n = \sum_{k=0}^{n} \binom{n}{k} h^k z^{n-k}$ we get

$$\begin{aligned}
\frac{F(z+h)-F(z)}{h} - f(z) &= \sum_{n=0}^{\infty} a_n \frac{(z+h)^n - z^n - hnz^{n-1}}{h} \\
&= \sum_{n=0}^{\infty} \frac{a_n}{h} \left[\sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k} \right] \\
&= \sum_{n=0}^{\infty} a_n h \left[\sum_{k=2}^{n} \binom{n}{k} h^{k-2} z^{n-k} \right] \\
&= \sum_{n=0}^{\infty} a_n h \left[\sum_{j=0}^{n-2} \binom{n}{j+2} h^{j} z^{n-2-j} \right]
\end{aligned}$$

where in last line we have taken $j = k-2$

Using $\binom{n}{j+2} \leq n(n-1)\binom{n-2}{j}$ we obtain

$$\frac{F(z+h)-F(z)}{h} - f(z) = |h| \sum_{n=0}^{\infty} n(n-1)|a_n| \left[\sum_{j=0}^{n-2} \begin{pmatrix} n-2 \\ j \end{pmatrix} |h|^j |z|^{n-2-j} \right]$$

$$= |h| \sum_{n=0}^{\infty} n(n-1)|a_n|(|z|+|h|)^{n-2} \qquad (1.2.36.)$$

We already know that series $\sum_{n=0}^{\infty} n(n-1)|a_n||z|^{n-2}$ converges for $|z| < R$

Now ☛ for $|z| < R$ and $h \to 0$ we have $|z| + |h| < R$ eventually

It thus follows $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} - f(z) = 0$ (1.2.37.)

whenever $|z| < R$

Exercise 1.2.2.

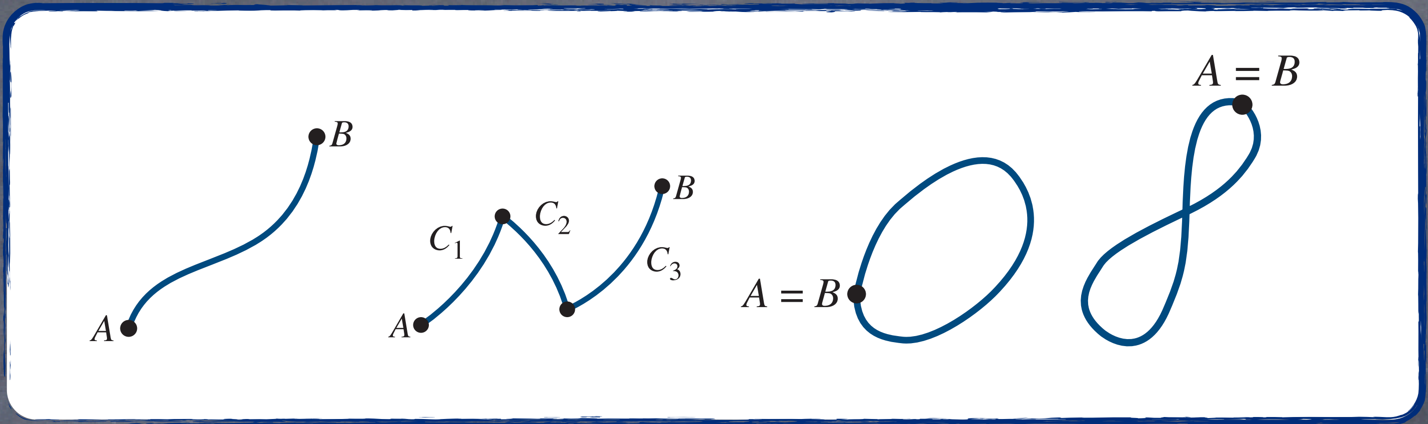
Using theorems 2.2.3. and 2.2.4. show that

$$e^z = \sum_{n=0}^{\infty} z^n/n!$$

Definition 1.3.1.

Suppose C is a curve parameterized by $x = f(t), y = g(t), a \leq t \leq b$ and A and B are points $(f(a), g(a))$ and $(f(b), g(b))$

We say that:

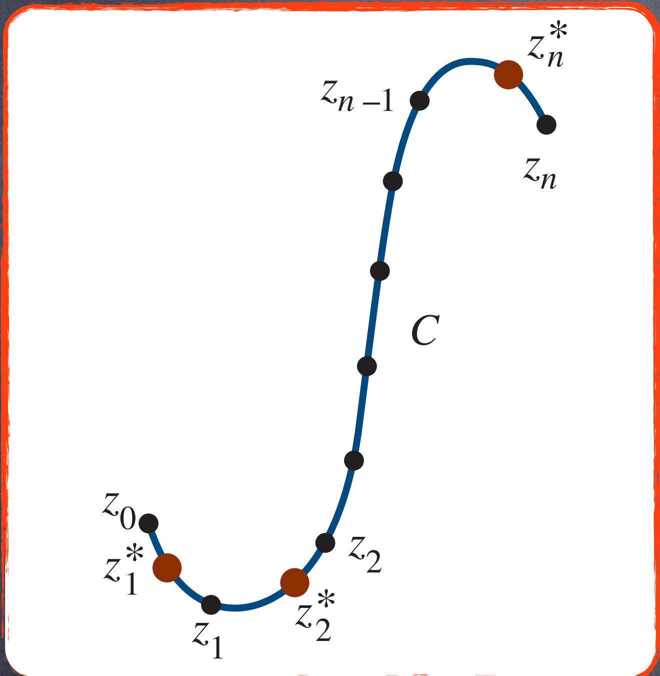


- C is a smooth curve if f' and g' are continuous on $[a, b]$ and not simultaneously zero on (a, b)
- C is piecewise smooth if $C = C_1 \cup C_2 \cup \cdots \cup C_n$
  $C_1, C_2, \ldots, C_n$ smooth curves
- C is a closed curve if $A = B$
- C is a simple closed curve if $A = B$
  and curve does not cross itself

Definition 1.3.2.

Let $f(z) = u(x,y) + iv(x,y)$ be defined at all points on a smooth curve C

Divide C into n sub-arcs $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b,]$



$x = x(t),\ y = y(t),\ a \le t \le b$ on each sub-arc

Let $\Delta z_k = z_k - z_{k-1}$, with $k = 1, 2, \ldots, n$

$\|P\|$ ☛ norm of partition (maximum value of $|\Delta z_k|$)

Choose sample point $z_k^* = x_k^* + iy_k^*$

Form sum $\sum_{k=1}^{n} f(z_k^*)\Delta z_k$ ☛ if $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$ exists and is independent of details of choosing points z_k^*

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k = \int_C f(z)\,dz \qquad (1.3.39)$$

Remark 1.3.1.

As an alternative ☛ contour integral may be defined by

$$\int_{z_1}^{z_2} f(z)\ dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y) + iv(x,y)][dx + idy]$$
$$= \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y)dx - v(x,y)dy] + i \int_{(x_1,y_1)}^{(x_2,y_2)} [v(x,y)dx + u(x,y)dy]$$

with path joining (x_1, y_1) and (x_2, y_2) specified

This reduces complex integral to complex sum of real integrals

Theorem 1.3.1. [ML- inequality]

If f is continuous on a smooth curve C

and if $|f(z)| \leq M$ for all z on C

then $|\int_C f(z)dz| \leq ML$ where ☛ L is length of C

Proof.

using $|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| \cdots + |z_n|$

$$\left|\sum_{k=1}^{n} f(z_k^*)\Delta z_k\right| \leq \sum_{k=1}^{n} |f(z_k^*)|\ |\Delta z_k| \leq M \sum_{k=1}^{n} |\Delta z_k| \quad (1.3.40.)$$

$|\Delta z_k|$ is length of chord joining points z_k and z_{k-1}

Since sum of lengths of chords cannot be greater than C length

(2.3.40.) becomes $\left|\sum_{k=1}^{n} f(z_k^*)\Delta z_k\right| \leq ML$

If $\|P\| \to 0$ ☛ last inequality yields $|\int_C f(z)dz| \leq ML$


TO BE
CONTINUED....