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Problems set # 7 **Physics 307** November 2, 2016

Ordinary Differential Equations III

1. A particle of mass m is at rest at the end of a spring (force constant $= k$). At $t = 0$ a constant force is applied to the mass and acts for a time t_0. Show that after the force is removed, the displacement of the mass from its equilibrium position $x = x_0$, is:

$$x - x_0 = F_0/k\,[\cos\omega_0(t - t_0) - \cos\omega_0 t],$$

where $\omega_0^2 = k/m$. Neglect friction effects!

2. Consider the boundary value problem $u'' + \lambda u = 0$, with $u(0) - u'(0) = 0$, $u(1) + u'(1) = 0$.
(i) Using the Rayleigh quotient, $\frac{\langle u, Lu\rangle}{\langle u,u\rangle} = \frac{\int_a^b u(x)\left\{-\frac{d}{dx}[p(x)u'(x)] + q(x)u(x)\right\}dx}{\int_a^b u^2(x)dx}$ show that $\lambda \geq 0$. Why is $\lambda > 0$?
(ii) Show that $\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}$.
(iii) Determine the eigenvalues graphically. Estimate the large eigenvalues.

3. Find the Green function and give an expression for the solution of the following (inhomogeneous) Sturm-Liouville problems:
(i) $-u'' = f(x)$, with $u(0) = u'(1) = 0$;
(ii) $-(x^2u')' + 2u = f(x)$, with $2u(1) + u'(1) = u'(2) = 0$. For the latter, express $G(x, x')$ as a piecewise defined rational function.

4. Let $L[u(x)] = -(x^2u')'$, $x \in [1, 2]$ be a Sturm-Liouville operator with domain $D = \{u \in \mathcal{C}^2[1, 2] : u(1) = u(2) = 0\}$.
(i) Express its Green function as a (factored) rational function.
(ii) Find the inverse operator $L^{-1}[u(x)]$.

5. Consider the inhomogeneous Sturm-Liouville problem

$$\frac{d^2u}{dx^2} + ku = f(x),$$

with $k > 0$ and $u(0) = u(\ell) = 0$.
(i) Determine the values of ℓ for which the problem is not singular.
(ii) Find the Green function for such values of ℓ and give an expression for the solution.