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Problems set # 4 **Physics 307** October 5, 2016

Elements of Linear Algebra

1. Let V be a vector space over F and let T be a linear transformation of the vector space V to itself. A nonzero element $\boldsymbol{x} \in V$ satisfying $T(\boldsymbol{x}) = \lambda \boldsymbol{x}$ for some $\lambda \in F$ is called an eigenvector of T, with eigenvalue λ. Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of T with eigenvalue λ together with $\mathbf{0}$ forms a subspace of V, that is, a subset of the vector space V that is closed under addition and scalar multiplication.

2. *(i)* Show that $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions.
(ii) Find all unit vectors lying in span$\{(3,4)\}$.

3. A matrix $\mathbb{A} \in \mathbb{C}^{n\times n}$ is nilpotent if $\mathbb{A}^k = 0$ for some integer $k > 0$. Prove that the only eigenvalue of a nilpotent matrix is zero.

4. *(i)* Determine whether the function $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x^2, y)$ is linear?
(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,0,0) = (2,4,-1), \quad T(0,1,0) = (1,3,-2), \quad T(0,0,1) = (0,-2,2);$$

compute $T(-2,4,-1)$.
(iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x_1,x_2,x_3) = (2x_1 + x_2, 2x_2 - 3x_1, x_1 - x_3), \quad \boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3;$$

compute $T(-4,-5,1)$.
(iv) Let $T : \mathbb{R}^5 \to \mathbb{R}^2$ be a linear transformation $T(\boldsymbol{x}) = \mathbb{A}\boldsymbol{x}$, with

$$\mathbb{A} = \begin{pmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{pmatrix};$$

compute $T(1,0,-1,3,0)$.
(v) Let $T(x,y,z) = (3x - 2y + z, 2x - 3y, y - 4z)$. Write down the matrix representation of T in the standard basis and use it to find $T(2,-1,-1)$.
(vi) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a_1,a_2,a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. Prove that T is an isomorphism and find T^{-1}.

5. *(i)* Show that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation by a fixed angle θ, then $T(x,y) = \mathbb{A}\boldsymbol{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.
(ii) Let T be the counterclockwise rotation in $\mathbb{R}^2$ by an angle 120°, write down the matrix of T

and compute $T(2,2)$.
(iii) Prove that if θ is not an integer multiple of π there does not exist a real valued matrix $\mathbb{B}$ such that $\mathbb{B}^{-1}\mathbb{A}\mathbb{B}$ is a diagonal matrix.

6. Let $\boldsymbol{x} \in \mathbb{R}^n$ be a vector. Then, for $\boldsymbol{y} \in \mathbb{R}^n$, define $\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y}) = \frac{\boldsymbol{x}\,.\,\boldsymbol{y}}{\|\boldsymbol{x}\|^2}\,\boldsymbol{x}$. The point of such projections is that any vector $\boldsymbol{y} \in \mathbb{R}^n$ can be written uniquely as a sum of a vector along $\boldsymbol{x}$ and another one perpendicular to $\boldsymbol{x}$: $\boldsymbol{y} = \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y}) + [y - \mathrm{proj}_{\boldsymbol{x}}[\boldsymbol{y}]]$. It is easy to check that $[y - \mathrm{proj}_{\boldsymbol{x}}[\boldsymbol{y}]] \perp \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})$.
(i) Show that $\mathrm{proj}_{\boldsymbol{x}} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.
(ii) Let T be the projection on to the vector $\boldsymbol{x} = (1,-5) \in \mathbb{R}^2 : T(\boldsymbol{y}) = \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})$; find the matrix representation in the standard basis and compute $T(2,3)$.

7. Show that if $\mathbb{A} = \begin{pmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{pmatrix}$ then $\lim_{n\to\infty} \mathbb{A}^n = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$.

8. *(i)* Show that Hermitian matrices satisfy the following properties $(\mathbb{A}\mathbb{B})^\dagger = \mathbb{B}^\dagger\mathbb{A}^\dagger$.
(ii) Prove that the inverse of a Hermitian matrix is again a Hermitian matrix.

9. Consider a 3×3 real symmetric matrix with determinant 6. Assume $\boldsymbol{x_1} = (1,2,3)$ and $\boldsymbol{x_2} = (0,3,-2)$ are eigenvectors with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.
(i) Give an eigenvector of the form $\boldsymbol{x_3} = (1, x_2, x_3)$ for some real x_2, x_3 which is linearly independent of the two vectors above.
(ii) What is the eigenvalue of this eigenvector.

10. *(i)* Find the eigenvalues and eigenvectors of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Show that the Pauli matrices obey the following commutation and anticommutation relations: $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$.
(iii) Show that $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ are linearly independent.
(iv) Prove that $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis in 2×2 matrix space, by showing that any arbitrary matrix

$$\mathbb{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

can be written on the form $\mathbb{M} = a_0\mathbb{1} + \vec{a}\,.\,\vec{\sigma}$, where $a_0 = \frac{1}{2}\mathrm{Tr}\,(M)$, $\vec{a} = \frac{1}{2}\mathrm{Tr}\,(\mathbb{M}\vec{\sigma})$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector.
(v) Let $\vec{v}$ be any real, three dimensional unit vector and θ a real number. Show that

$$\exp(i\theta\,\vec{v}\,.\,\vec{\sigma}) = \mathbb{1}\cos\theta + i\vec{v}\,.\,\vec{\sigma}\,\sin\theta,$$

where $\vec{v}\,.\,\vec{\sigma} \equiv \sum_{i=1}^{3} v_i\sigma_i$, with σ_i the Pauli matrices.