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Problems set # 10 **Physics 501** November 22, 2016

Partial Differential Equations I

1. Consider the wave equation for a vibrating rectangular membrane ($0 < x < L$, $0 < y < H$), $u_{tt} = c^2(u_{xx} + u_{yy})$, subject to the initial conditions $u(x, y, 0) = 0$ and $u_t(x, y, 0) = f(x, y)$. Solve the initial-boundary value problem if $u_x(0, y, t) = 0$, $u_x(L, y, t) = 0$, $u_y(x, 0, t) = 0$, $u_y(x, H, t) = 0$.

2. Use the Fourier transform to derive d'Alembert's solution of the wave equation for $-\infty < x < \infty$; $u_{tt} = c^2 u_{xx}$, $u(x, 0) = f(x)$, $u_t(x, 0) = q(x)$. [*Hint*: First show that $\frac{1}{k}\mathscr{F}[q] = i\mathscr{F}[\int_{-\infty}^{x} q(\zeta)\, d\zeta$.]

3. Consider the wave equation for a string of length L with initial displacement $f(x)$ and zero initial velocity $q(x)$, *i.e.* $u_{tt} = c^2 u_{xx}$, $u(0, t) = u(L, t) = 0$, $u(x, 0) = f(x)$, $u_t(x, 0) = 0$. Using the result of exercise 2 it is easily seen that the separation of variables solution is given by $u(x, t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct/L)\sin(n\pi x/L)$, where $B_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\, dx$. Starting with this formula and the trigonometric identity $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\beta-\alpha)]$, break the series for $u(x, t)$ into two series, and arrive at the d'Alembert solution $u(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$.

4. Use separation of variables to find the solution of the 2-dimensional Helmholtz equation $\nabla^2 u = -k^2 u$ that remains bounded at all points in the disk $0 \leq r \leq 1$, that is periodic with period 2π, and satisfies the boundary conditions $u(1, \theta) = f(\theta)$. [*Hint:* In polar coordinates the Helmholtz equation becomes $u_{rr} + u_r/r + u_{\theta\theta}/r^2 = -k^2 u$.]

5. Consider the wavefunction of a free particle of mass m, which has a fixed oscillation in time, $\Psi(\boldsymbol{x}, t) = \psi(\boldsymbol{x})\, e^{i\omega t}$. Entering the functional form of Ψ into Schrödinger equation gives us $\omega\hbar\psi e^{i\omega t} = -\frac{\hbar}{2m}\nabla^2\psi e^{i\omega t}$. Noting that $\omega\hbar = E$, the energy in the quantum setting, after dividing by e^{ikt} we obtain, $-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$. The function ψ is called a stationary state or orbital and $|\psi(\boldsymbol{x})|^2$ represents the probability distribution of the spatial location for a particle at a fixed energy E. In general only certain quantized values of E are possible and the differential equation becomes a Helmholtz eigenvalue problem. Calculate the eigenfunctions and energy levels for a free particle, enclosed in a box with edges of lengths a, b, and c. [Hint: The presence of the box (because of continuity) requires the wave function to vanish at the edges.]

6. The vibrations of an idealized circular drumhead – essentially an elastic membrane of uniform thickness attached to a rigid circular frame – are solutions of the wave equation with zero boundary conditions. Find the Green function for the equation that governs the vibration of the drumhead, that is, solve the 2-dimensional Helmholtz equation, $G_{xx} + G_{yy} + k^2 G = -\delta(x - x')\,\delta(y - y')$, in the region $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, which cancels on the border. Due to the circular geometry, it will be convenient to use polar coordinates where the previous equation reads $G_{rr} + \frac{1}{r}\, G_r + \frac{1}{r^2}\, G_{\theta\theta} + k^2 G = -\delta(r - r')\,\delta(\theta - \theta')\,\frac{1}{r}$. [*Hint*: Expand G in Fourier series.]

9. Find the free-space Helmholtz Green function in the outer region $r \geq a$, $0 \leq \theta \leq 2\pi$. Consider the solution that cancels at $r = a$ and satisfies the Sommerfeld radiation condition, *i.e.*

$$\lim_{r \to \infty} r^{1/2}(G_r - ikG) = 0.$$