Modern Physics

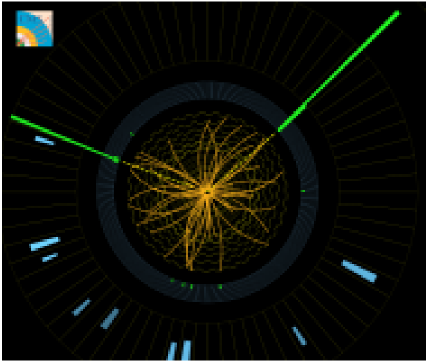
Luis A. Anchordoqui

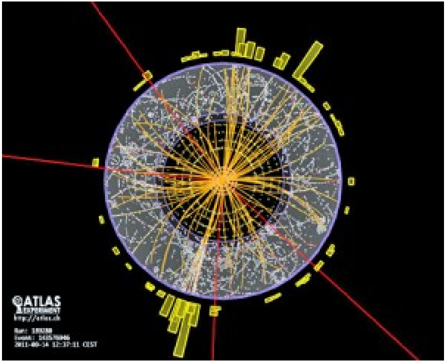
Department of Physics and Astronomy
Lehman College, City University of New York

Lesson VII
October 5, 2023

Table of Contents

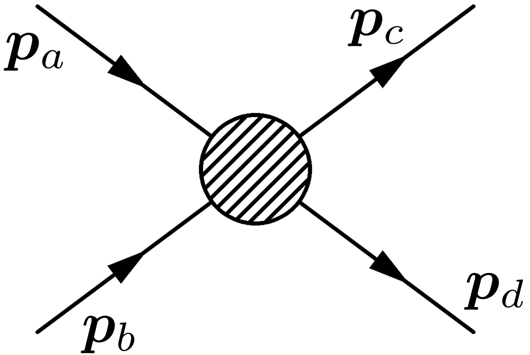
1. Particle Decay and Collisions
   - Two-body scattering
   - Threshold energy
   - Transverse mass, rapidity, and pseudorapidity




ATLAS

- In high energy physics☞ cross sections and decay rates are written using kinematic variables that are relativistic invariants
- For any "two particle to two particle" process $ab \to cd$ we have at our disposal 4-momenta associated with each particle
- Invariant variables are six scalar products:

$$p_a \cdot p_b,\ p_a \cdot p_c,\ p_a \cdot p_d,\ p_b \cdot p_c,\ p_b \cdot p_d,\ p_c \cdot p_d$$



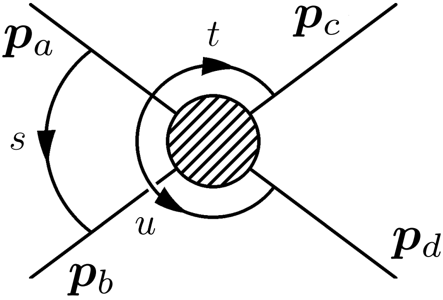
- Rather than these ☞ use Mandelstam variables

$$s = c^2(\boldsymbol{p}_a + \boldsymbol{p}_b)^2 \qquad t = c^2(\boldsymbol{p}_a - \boldsymbol{p}_c)^2 \qquad u = c^2(\boldsymbol{p}_a - \boldsymbol{p}_d)^2$$

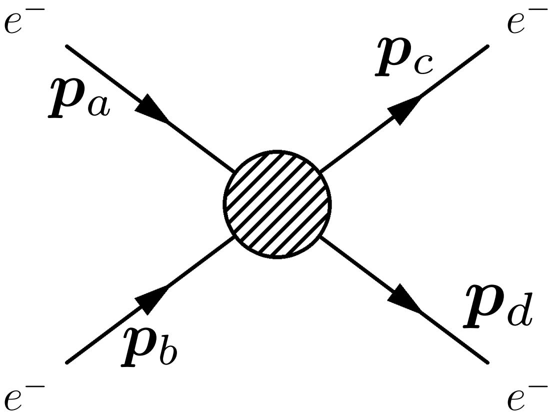
- Because $\boldsymbol{p}_i^2 = m_i^2 c^2$ (with $i = a, b, c, d$) and $\boldsymbol{p}_a + \boldsymbol{p}_b = \boldsymbol{p}_c + \boldsymbol{p}_d$

$$s + t + u = \sum_i m_i^2 c^4 + c^2 \left[2\boldsymbol{p}_a^2 + 2\boldsymbol{p}_a.(\boldsymbol{p}_b - \boldsymbol{p}_c - \boldsymbol{p}_d)\right] = \sum_i m_i^2 c^4$$

Only 2 of 3 variables are independent



Møller scattering



Møller scattering scattering in CM frame

4-momenta are

$$\boldsymbol{p}_a = (E/c, \vec{p}_i),\ \boldsymbol{p}_b = (E/c, -\vec{p}_i),\ \boldsymbol{p}_c = (E/c, \vec{p}_f),\ \boldsymbol{p}_d = (E/c, -\vec{p}_f)$$

$E = (p^2c^2 + m_e^2c^4)^{1/2}$ ☞ mass-shell condition

$$\begin{aligned} s &= 4(p^2c^2 + m_e^2c^4) \\ t &= -c^2(\vec{p}_i - \vec{p}_f)^2 = -2p^2c^2(1-\cos\theta^*) \\ u &= -c^2(\vec{p}_i + \vec{p}_f)^2 = -2p^2c^2(1+\cos\theta^*) \end{aligned}$$

θ^* ☞ scattering angle $\Rightarrow \vec{p}_i \cdot \vec{p}_f = p^2 \cos\theta^*$

- As $p^2 \geq 0$ ☞ $s \geq 4m_e^2c^4$
- Since $-1 \leq \cos\theta^* \leq 1$ ☞ $t \leq 0$ and $u \leq 0$
- $t = 0$ ($u = 0$) corresponds to forward (backward) scattering

- In CM frame for reaction $ab \to cd$
  - $s \equiv$ square CM energy $E_{\rm CM}^2$ ☞ $E_{\rm CM} = E_a + E_b$
  - $t \equiv$ square of momentum transfer between particles a and c
  - $u \equiv$ square of momentum transfer between particles a and d (not independent variable)
- This is called s-channel process
- In s-channel ☞ s is positive while t and u are negatives
- The process is elastic if $m_a = m_c$ and $m_b = m_d$

Take a closer look at general process $ab \to cd$

CM frame is defined by ☞ $\vec{p}_a + \vec{p}_b = \vec{0} = \vec{p}_c + \vec{p}_d$

4-momenta are

$$\boldsymbol{p}_a = (E_a^*/c, \vec{p}_i),\ \boldsymbol{p}_b = (E_b^*/c, -\vec{p}_i),\ \boldsymbol{p}_c = (E_c^*/c, \vec{p}_f),\ \boldsymbol{p}_d = (E_d^*/c, -\vec{p}_f)$$

On-shell conditions lead to

$$E_a^* = \sqrt{\vec{p}_i^{\,2}c^2 + m_a^2c^4} \qquad E_b^* = \sqrt{\vec{p}_i^{\,2}c^2 + m_b^2c^4}$$

$$E_c^* = \sqrt{\vec{p}_f^{\,2}c^2 + m_c^2c^4} \qquad E_d^* = \sqrt{\vec{p}_f^{\,2}c^2 + m_d^2c^4}$$

After some algebra ...

we express $E^*_{a,b,c,d}$, $|\vec{p}_i|$, $|\vec{p}_f|$ in terms of $s = c^2(\boldsymbol{p}_a + \boldsymbol{p}_b)^2 = (E_a^* + E_b^*)^2$

$$E^*_{a,c} = \frac{1}{2\sqrt{s}}\left(s + m^2_{a,c}c^4 - m^2_{b,d}c^4\right) \qquad E^*_{b,d} = \frac{1}{2\sqrt{s}}\left(s + m^2_{b,d}c^4 - m^2_{a,c}c^4\right)$$

$$p_i^2c^2 = E_a^{*2} - m_a^2c^4 = \frac{1}{4s}\lambda(s, m_a^2c^4, m_b^2c^4) \qquad p_f^2c^2 = \frac{1}{4s}\lambda(s, m_c^2c^4, m_d^2c^4)$$

Källen (triangle) function ☞ $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$

$$\begin{aligned}\lambda(a,b,c) &= a^2+b^2+c^2-2ab-2ac-2bc \\ &= \left[a-(\sqrt{b}+\sqrt{c})^2\right]\left[a-(\sqrt{b}-\sqrt{c})^2\right] \\ &= a^2-2a(b+c)+(b-c)^2\end{aligned}$$

- Properties of Källen function
  - λ is symmetric under $a \leftrightarrow b \leftrightarrow c$
  - $\lambda(a,b,c) \to a^2$, for $a \gg b, c$
- This enables to determine properties of scattering processes
- High energy limit ☞ $s \gg m^2_{a,b,c,d} c^4$
  $E^*_{a,b,c,d}$, $|\vec{p}_i|$, and $|\vec{p}_f|$ simplify because of asymptotic behavior of λ

$$E^*_a = E^*_b = E^*_c = E^*_d = c|\vec{p}_i| = c|\vec{p}_f| = \sqrt{s}/2$$

In CM frame scattering angle defined by

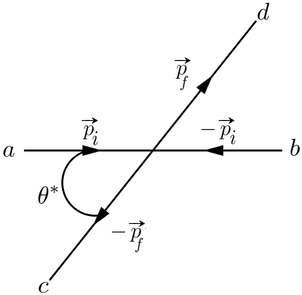
$$\vec{p}_i \cdot \vec{p}_f = |\vec{p}_i| \cdot |\vec{p}_f| \cos\theta^*$$

using

$$\boldsymbol{p}_a \cdot \boldsymbol{p}_c = E_a^* E_c^* / c^2 - |\vec{p}_a^{\,*}||\vec{p}_c^{\,*}| \cos\theta^*$$

and

$$\begin{aligned} t &= c^2(\boldsymbol{p}_a - \boldsymbol{p}_c)^2 = (m_a^2 + m_c^2)c^4 - 2c^2 \boldsymbol{p}_a \cdot \boldsymbol{p}_c \\ &= (m_a^2 + m_c^2)c^4 - 2E_aE_c + 2c^2 \vec{p}_a \cdot \vec{p}_c \\ &= (m_a^2 + m_c^2)c^4 - 2E_aE_c + 2c^2 p_i p_f \cos\theta^* = c^2(\boldsymbol{p}_b - \boldsymbol{p}_d)^2 \end{aligned}$$



we write scattering angle as function of s, t, $m^2_{a,b,c,d}$

$$\cos\theta^* = \frac{s(t-u) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)c^8}{\sqrt{\lambda(s, m_a^2c^4, m_b^2c^4)}\sqrt{\lambda(s, m_c^2c^4, m_d^2c^4)}}$$

This means that $2 \to 2$ scattering is described by two variables:

$(\sqrt{s}, \theta^*)$ or else $(\sqrt{s}, t)$

- One way to create exotic heavy particle X
  is to arrange collision between two lighter particles

$$a + b \to X + d + e + \cdots + g$$

$d, e, \ldots, g$ are other possible particles produced in reaction

- In all such cases ☞ theoretical minimum expenditure of energy occurs when all end-products are mutually at rest
- Consider projectile a and stationary target b with $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$
- If emergent particles have 4-momenta $\boldsymbol{p}_i$ $(i = 1, 2, \cdots)$

$$\boldsymbol{p}_a + \boldsymbol{p}_b = \boldsymbol{p}_X + \boldsymbol{p}_d + \boldsymbol{p}_e \cdots + \boldsymbol{p}_g = \sum_i \boldsymbol{p}_i \tag{1}$$

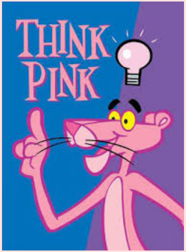
Interlude

- Consider two particles with $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ and relative speed v_{ab} (v_{ab} ☞ speed of one in rest-frame of the other)

$$\boldsymbol{p}_a \cdot \boldsymbol{p}_b = m_a E_b = m_b E_a = c^2 \gamma(v_{ab}) m_a m_b \tag{2}$$

m_a ☞ rest-mass of first particle
E_b ☞ energy of second particle in rest-frame of first

- To verify (2) ☞



evaluate $\boldsymbol{p}_a \cdot \boldsymbol{p}_b$ in rest-frame of either particle

- Squaring (1)

$$m_a^2 + m_b^2 + \frac{2m_b E_a}{c^2} = \sum_i m_i^2 + 2 \sum_{(i<j)} m_i m_j \gamma(v_{ij}) \tag{3}$$

- All the masses in (3) are fixed
- Only variable on l.h.s. is E_a ☞ energy of projectile relative to lab
- Minimum of r.h.s. when all Lorentz factors are unity
  there is no relative motion between any of the outgoing particles
- Threshold energy of projectile

$$E_a = \frac{c^2}{2m_b} \left[\left(\sum_i m_i \right)^2 - m_a^2 - m_b^2 \right] \tag{4}$$

- (4) also applies if projectile is $\underbrace{\gamma}_{\text{photon}}$ getting absorbed in collision

Example

$$pp \to pp\pi^0$$

$$E_p - m_p c^2 = c^2 \left(2m_\pi + \frac{m_\pi^2}{2m_p} \right)$$

- Efficiency k ☞ ratio of π rest energy to p kinetic energy

$$k = m_\pi \left(2m_\pi + \frac{m_\pi^2}{2m_p} \right)^{-1} = \frac{2}{4 + (m_\pi/m_p)}$$

- Efficiency ☞ always less than 50%
- For $pp \to pp\pi^0$☞ $m_\pi/m_p \approx 0.14$ and $k \approx 48\%$
- If $m_X \gg m_e, m_g, \cdots$

$$k \approx 2m_b/m_X$$

Example

- $e^+e^- \to J/\psi$ ☞ $k \sim 1/1850$
- Colliding beams to the rescue ☞ almost 100% efficiency
- Both target and projectile particles are accelerated to high energy
- No "waste" kinetic energy need be present after collision since there was no net momentum going in
- For $m_b = m_e \approx 0.5 \text{ MeV}/c^2$ and $m_{J/\psi} \approx 3100 \text{ MeV}/c$

$$E_{\rm CM} \approx m_{J/\psi} c^2 \approx 3100 \text{ MeV}$$

whereas

$$E_{\rm lab} \approx \frac{m_{J/\psi}^2 c^2}{2 m_e} = 9600000 \text{ MeV}$$

- Introduce invariants of common use in collider physics
  which derive from the fact that
  velocities of colliding particles are along beam axis

- Invariants with respect to observers
  who are Lorentz boosted with respect to the z-axis

- What is special about these observers?

- Accelerators collide particles
  whose momentum is not equal and opposite
  but whose directions are down a common beam z-axis

- CM frame is moving at some velocity down z-axis
  so you will often wish to study physics in this frame

- However ☞ if you are stuck in lab frame
  you are boosted with some velocity v_z with respect to this frame
  and the direction of the boost is parallel to the beam axis

Rapidity

$$y = \frac{1}{2} \ln \left(\frac{E + p_z c}{E - p_z c} \right)$$

Why would you want to define such a quantity?

- Suppose we are dealing with high energy product of a collision (highly relativistic regime)
- If particle is directed in x-y $\perp$ to beam direction
  p_z will be small ☞ $y \to 0$
- If particle is directed down beam axis ☞ say in $+z$ direction
  $E \simeq p_z c$ ☞ $y \to +\infty$.
- Similarly ☞ if particle is travelling down beam axis in $-z$ direction
  $E \simeq -p_z c$ ☞ $y \to -\infty$
- Rapidity related to:
  angle between x-y plane and direction of secondary product

Transverse mass

- E and p_z can separately be expressed as functions of rapidity
- Rewrite energy-momentum-mass relation

$$E^2 = M_T^2 c^4 + p_z^2 c^2 \tag{5}$$

in terms of transverse mass

$$M_T^2 c^4 = p_x^2 c^2 + p_y^2 c^2 + m^2 c^4$$

- x and y components of momentum and particle mass are all invariant with respect to boosts parallel to z-axis
- Rewriting (5) as $\left(\frac{E}{M_T c^2}\right)^2 - \left(\frac{p_z}{M_T c}\right)^2 = 1$ and comparing with $\cosh^2 y - \sinh^2 y = 1$

$$\boldsymbol{p} \equiv (E/c = M_T c \cosh y,\ p_x,\ p_y,\ p_z = M_T c \sinh y)$$

- Upon Lorentz boost parallel to beam axis with velocity $v = \beta c$ equation for transformation on rapidity is a particularly simple one

$$y' = y - \tanh^{-1}\beta$$

- Assume two secondaries have rapidities y_1 and y_2 measured in S
- Another observer moving along z-axis in S' measures y'_1 and y'_2
- Difference between rapidities

$$y'_1 - y'_2 = y_1 - \tanh^{-1}\beta - y_2 + \tanh^{-1}\beta = y_1 - y_2$$

  is invariant with respect to Lorentz boosts along z-axis
- Key variable in accelerator physics:
  Histograms binned in rapidity separation of events
  are undistorted by CM frame boosts parallel to beam axis
  as dependent variable is invariant wrt sub-class of Lorentz boosts

- Rapidity can be hard to measure for highly relativistic particles
  need to measure both energy and total momentum
- @ high rapidities where z component of momentum is large
  ☞ beam pipe can prevent measuring momentum precisely
- Define quantity that is almost same as rapidity
  but it is much easier to measure

$$y \simeq \eta = -\ln\left(\tan\frac{\theta}{2}\right)$$

θ ☞ angle made by particle trajectory with beam pipe

- Pseudorapidity η is particularly useful in hadron colliders
  ☞ composite nature of colliding protons means that interactions rarely have their CM frame coincident with detector rest frame