# Modern Physics

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Lesson VI
September 28, 2023

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- Newton's first law of motion holds in special relativistic mechanics as well as nonrelativistic mechanics
- In absence of forces ☞ body is at rest or moves in straight line at constant speed

$$\frac{d\boldsymbol{U}}{d\tau} = 0 \qquad (1)$$

$\boldsymbol{U}(\gamma c, \gamma \vec{u})$ ☞ (1) implies that $\vec{u}$ is constant in any inertial frame

- Objective of relativistic mechanics:
  ☞ introduce the analog of Newton's 2nd law

$$\vec{F} = m\vec{a} \qquad (2)$$

- There is nothing from which this law can be derived but plausibly it must satisfy certain properties:
  1. It must satisfy the principle of relativity
     i.e. ☞ take the same form in every inertial frame
  2. It must reduce to (1) when the force is zero
  3. It must reduce to (2) in any inertial frame
     when speed of particle is much less than speed of light

$$m\frac{d\boldsymbol{U}}{d\tau} = \boldsymbol{f} \tag{3}$$

- $m$ ☞ characterizes particle's inertial properties
- $\boldsymbol{f}$ ☞ 4-force.
- Using $d\boldsymbol{U}/d\tau = \boldsymbol{A}$ ☞ (3) can be rewritten in evocative form

$$\boldsymbol{f} = m\boldsymbol{A} \tag{4}$$

- This represents 4-equations ☞ but they are not all independent
- Normalization of the 4-velocity ☞ $U_\mu U^\mu = c^2$ implies

$$m\frac{d(\boldsymbol{U}\cdot\boldsymbol{U})}{d\tau} = 0 \Rightarrow \boldsymbol{f}\cdot\boldsymbol{U} = 0 \tag{5}$$

- (5) shows there are only 3 independent equations of motion
  – same number as in Newtonian mechanics –

4-momentum defined by

$$\boldsymbol{p} = m\,\boldsymbol{U} \tag{6}$$

equation of motion can be rewritten as

$$\frac{d\boldsymbol{p}}{d\tau} = \boldsymbol{f} \tag{7}$$

important property of 4-momentum ☞ invariant mass

$$p_\mu\, p^\mu = m^2 c^2 \tag{8}$$

- Components of 4-momentum related $\vec{u}$ according to

$$p^0 = \frac{mc}{\sqrt{1-u^2/c^2}} \quad \text{and} \quad \vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \tag{9}$$

- For small speeds $u \ll c$

$$p^0 = mc + \frac{1}{2}m\frac{u^2}{c} + \cdots \quad \text{and} \quad \vec{p} = m\vec{u} + \cdots \tag{10}$$

- $\vec{p}$ reduces to usual 3-momentum
- $p^0$ reduces to kinetic energy per units of $c$ plus mass in units of $c$
- $\boldsymbol{p}$ ☞ also called energy-momentum 4-vector

$$p^\mu = (E/c, \vec{p}) = (m\gamma c, m\gamma\vec{u}) \tag{11}$$

- mass is part of energy of relativistic particle

$$p_\mu\, p^\mu = m^2c^2 \Rightarrow E = (m^2c^4 + \vec{p}^{\,2}c^2)^{1/2} \tag{12}$$

- For particle at rest (12) reduces to ☞ $E = mc^2$

In particular inertial frame...

- Connection between relativistic equation of motion and Newton's laws can be made more explicit by defining 3-force

$$\frac{d\vec{p}}{dt} \equiv \vec{F} \tag{13}$$

- Same form as Newton's law but with relativistic expression for $\vec{p}$
- Only difference ☞ different relation momentum to velocity

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}/dt}{dt/d\tau} = \gamma\vec{F} \tag{14}$$

- 4-force acting on particle can be written in terms of 3-force

$$\boldsymbol{f} = (\gamma\vec{F}\cdot\vec{u}, \gamma\vec{F}) \tag{15}$$

- Time component of equation of motion

$$\frac{dE}{dt} = \vec{F}\cdot\vec{u} \tag{16}$$

familiar relation from Newtonian mechanics

- Time component of equation of motion consequence of other 3
- In terms of three force ☞ equations of motion
  take same form as they do in usual Newtonian mechanics
  but with relativistic expressions for energy and momentum
- For $v \ll c$ ☞ relativistic version of Newton's second law
  reduces to the familiar nonrelativistic form
- Newtonian mechanics
  is low-velocity approximation of relativistic mechanics

## Light rays

- Massless particles move at speed of light along null trajectories
- Proper time interval between any two points ☞ is zero
- Curve $x = ct$ could be written parametrically in term of $\lambda$

$$x^\mu = U^\mu \lambda \qquad \text{with} \qquad U^\mu = (c, c, 0, 0) \tag{17}$$

- $\boldsymbol{U}$ is a null vector ☞

$$\boldsymbol{U} \cdot \boldsymbol{U} = 0 \tag{18}$$

- Different parametrizations give different tangent 4-vectors but all have zero length
- With choice (17)

$$\frac{d\boldsymbol{U}}{d\lambda} = 0 \tag{19}$$

light ray equation of motion is same as for particle

- Basic law of collision mechanics ☞ conservation of 4-momentum:
  *Sum of 4-momenta of all particles going into point-collision is same as sum of 4-momenta of all those coming out*

$$\sum{}^{*} p_i = 0 \tag{20}$$

  $\sum^*$ ☞ sum that counts pre-collision terms positively and post-collision terms negatively

- For closed system ☞ conservation of total 4-momentum can be shown to be result of spacetime homogeneity

- Whether law is actually true must be decided by experiment

- Countless experiments have shown that total 4-momentum of isolated system is constant

## CM frame

- If we have a system of particles with 4-momenta $p_i$ subject to no forces except mutual collisions
- total 4-momentum $p_{\rm tot} = \sum p_i$ is timelike and future-pointing
- ☞ there exists an inertial frame $S$ in which spatial components of $p_{\rm tot}$ vanish

$S$ should be called center-of-momentum frame but is called CM frame

## Invariant mass

- Invariant mass of two particles with 4-momenta $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$
$$m_{ab}^2 c^2 = (\boldsymbol{p}_a + \boldsymbol{p}_b)^2$$
- Invariant mass useful to find mass of short-live unsatble particles from momenta of their observed decay products
- Consider $X \to a + b$ ☞ $\boldsymbol{p}_X = \boldsymbol{p}_a + \boldsymbol{p}_b$

$$\begin{aligned} m_X^2 c^2 &= (\boldsymbol{p}_a + \boldsymbol{p}_b)^2 = \boldsymbol{p}_a^2 + \boldsymbol{p}_b^2 + 2\boldsymbol{p}_a \cdot \boldsymbol{p}_b \\ &= m_a^2 c^2 + m_b^2 c^2 + 2E_a E_b / c^2 - 2\vec{p}_a \cdot \vec{p}_b \end{aligned} \tag{21}$$

- In high energy experiment
  3-momenta and masses of particles $a$ and $b$ must be measured
- For charged particles this requires a magnetic field
  and tracking of trajectory to measure bending
  as well as some means of particle identification
- One must also identify the vertex and measure the opening angle

- Consider decay process $X \to ab$
- In CM frame for $a$ and $b$ mother particle $X$ is at rest
- 4-momenta

$$\boldsymbol{p}_X = (Mc, 0, 0, 0) \quad \boldsymbol{p}_a = (E_a/c, \vec{p}_a) \quad \boldsymbol{p}_b = (E_b/c, \vec{p}_b) \tag{22}$$

- Conservation of 4-momentum requires:

$$\boldsymbol{p}_X = \boldsymbol{p}_a + \boldsymbol{p}_b \text{ ☞ } \vec{p}_a = -\vec{p}_b$$

- Omitting subscript on 4-momenta ☞ energy conservation reads

$$E_a + E_b = \sqrt{m_a^2 c^4 + p^2 c^2} + \sqrt{m_b^2 c^4 + p^2 c^2} = Mc^2 \tag{23}$$

- Solving (23) for $p$

$$p = c \frac{\sqrt{[M^2 - (m_a - m_b)^2][M^2 - (m_a + m_b)^2]}}{2M} \tag{24}$$

- Immediate consequence

$$M \geq m_a + m_b \tag{25}$$

$X$ decays only if mass exceeds sum of decay products masses

- Conversely
  ☞ if particle has mass exceeding masses of two other particles
  particle is unstable and decays
  unless decay is forbidden by some conservation law
  e.g. conservation of charge, momentum, and angular momentum
- Momenta of daughter particles and energies fixed by 3 masses:
  from energy conservation (23) ☞ $E_b = \sqrt{E_a^2 - m_a^2 c^4 + m_b^2 c^4}$
  solve to get

$$E_a = \frac{1}{2M}(M^2 + m_a^2 - m_b^2)c^2 \qquad (26)$$

  similarly

$$E_b = \frac{1}{2M}(M^2 + m_b^2 - m_a^2)c^2 \qquad (27)$$

- No preferred direction in which the daughter particles travel
  decay is said to be *isotropic*
  ☞ daughter particles travelling *back-to-back* in $X$ rest frame

- Of interest is also two-body decay of unstable particles in flight
- In-flight decays ☞ only way to measure mass of neutral particle
- Take $z$-axis along direction of flight of mother particle

$$\boldsymbol{p}_X = (E/c, 0, 0, p) \quad \boldsymbol{p}_a = (E_a/c, \vec{p}_{a\perp}, p_{az}) \quad \boldsymbol{p}_b = (E_b/c, \vec{p}_{b\perp}, p_{bz})$$

- By momentum conservation ☞ transverse momentum vectors

$$\vec{p}_\perp \equiv \vec{p}_{a\perp} = -\vec{p}_{b\perp} \tag{28}$$

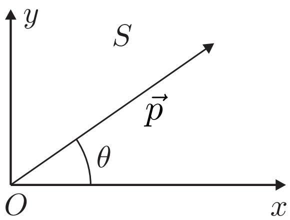
- Energies and $z$ components of particle momenta
  related to those in the CM frame by a Lorentz boost
  with a boost velocity equal to the speed of the mother particle

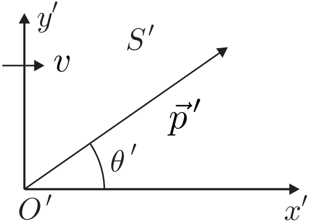
$$\begin{aligned} E_a/c &= \gamma(E_a^*/c + \beta p_{az}^*) \\ p_{az} &= \gamma(p_{az}^* + \beta E_a^*/c) \\ \vec{p}_{a\perp} &= \vec{p}_{a\perp}^* \end{aligned}$$

$\beta = pc/E$ and $\gamma = E/(Mc^2)$

similarly for particle $b$

- This completely solves problem
  e.g. ☞ we can find angles which daughter particles
  make with $z$-axis and with each other as functions of $\boldsymbol{p}_X$




- In $S$ we have

$$p^\mu = (E/c, p\cos\theta, p\sin\theta, 0) \tag{29}$$

  in $S'$ it follows that

$$p'^\mu = (E'/c, p'\cos\theta', p'\sin\theta', 0) \tag{30}$$

## Applying $S \to S'$ Lorentz transformation

$$\begin{aligned} p' \cos\theta' &= \gamma^* (p\cos\theta - \beta^* E/c) \\ p' \sin\theta' &= p \sin\theta \end{aligned} \tag{31}$$

so

$$\tan\theta' = \frac{p\sin\theta}{\gamma^*(p\cos\theta - \beta^* E/c)} \tag{32}$$

or

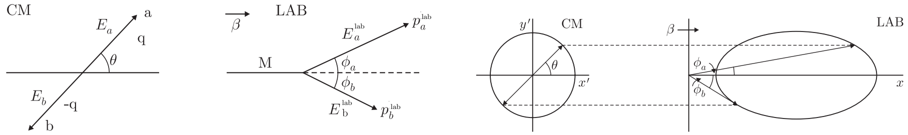
$$\tan\theta' = \frac{\sin\theta}{\gamma^*(\cos\theta - \beta^*/\beta)} \tag{33}$$

$\beta^* = v/c$ ☞ velocity of $S'$ wrt $S$ and $\beta = pc/E$ ☞ velocity of particle in $S$

Inverse relation is found to be

$$\tan\theta = \frac{\sin\theta'}{\gamma^*(\cos\theta' + \beta^*/\beta')} \tag{34}$$

$\beta' = p'c/E'$ ☞ velocity of particle in $S'$




## Example

- Mother particle of mass $M$ is traveling with velocity $\beta = |pc|/E$
- In the CM frame ☞ particle $a$ has energy $E_a$ and momentum $\vec{q}$ @ angle $\theta$ with respect to $x'$-axis
- Particle momenta in the lab frame

$$
\begin{aligned}
p_a^{\rm lab} \cos \phi_a &= \gamma(q \cos \theta + \beta E_a/c) \\
p_b^{\rm lab} \cos \phi_b &= \gamma(-q \cos \theta + \beta E_b/c)
\end{aligned}
\tag{35}
$$

- Use inverse Lorentz transformation to obtain variables in CM frame from measured parameters in lab

- 2nd approach ☞ start from energy-momentum conservation

$$E = E_a + E_b = \sqrt{m_a^2 c^4 + p_a^2 c^2} + \sqrt{m_b^2 c^4 + p_b^2 c^2} \tag{36}$$

$$\vec{p} = \vec{p}_a + \vec{p}_b \tag{37}$$

- Substituting in (36) $p_b^2$ by $(\vec{p} - \vec{p}_a)^2$

$$p_a = \frac{(M^2 + m_a^2 - m_b^2)c^2 \; p \cos\theta_a \pm 2E \sqrt{M^2 p^{*2} - m_a^2 p^2 \sin^2\theta_a}}{2(M^2 c^2 + p^2 \sin^2\theta_a)}$$

- By demanding $p_a$ to be real ☞ $M^2 p^{*2} - m_a^2 p^2 \sin^2\theta_a \geq 0$
- This condition is satisfied for all angles $\theta_a$ if $Mp^*/(m_a p) > 1$
  negative sign must be rejected ☞ unphysical $p_a < 0$ for $\theta_a > \pi/2$
- If $Mp^*/(m_a p) < 1$
  ☞ region of parameter space in which both signs must be kept:
  for each value of $\theta_a < \theta_{a,\max}$ there are two values of $p_a$
  and correspondingly also two values of $p_b$