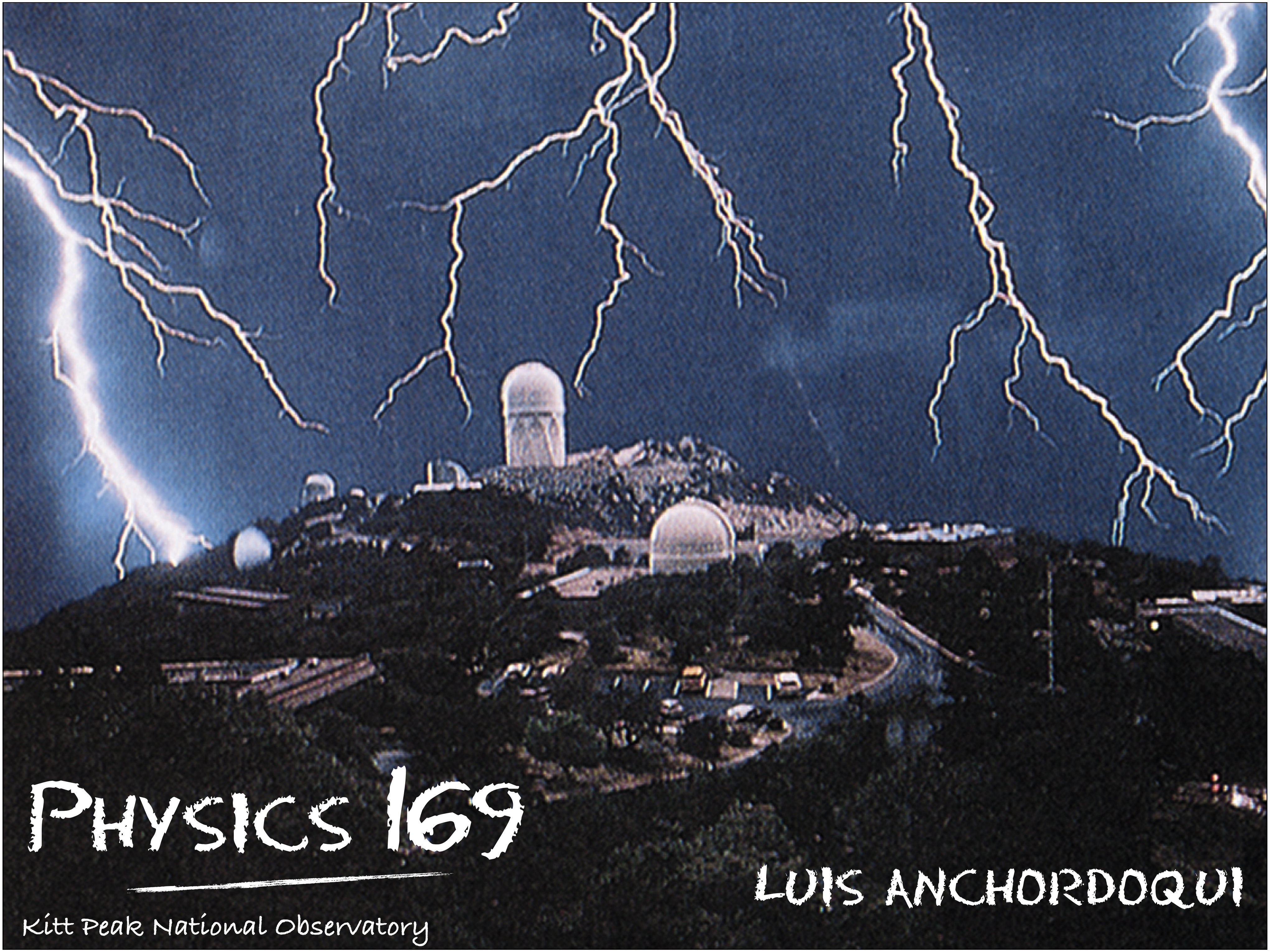

Physics 169
Luis Anchordoqui
Kitt Peak National Observatory

9.1 Inductance

An inductor stores energy in magnetic field

just as a capacitor stores energy in **electric field**

A changing **B-field will lead to an induced emf in a circuit**

Question

If a circuit generates a changing magnetic field

does it lead to an induced emf in same circuit?

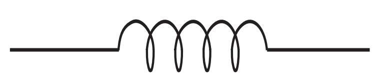
YES! Self-Inductance

Inductance L of any current element is

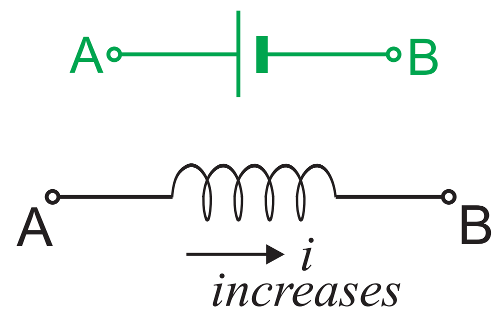
$$\mathcal{E}_L = \Delta V_L = -L\frac{di}{dt}$$

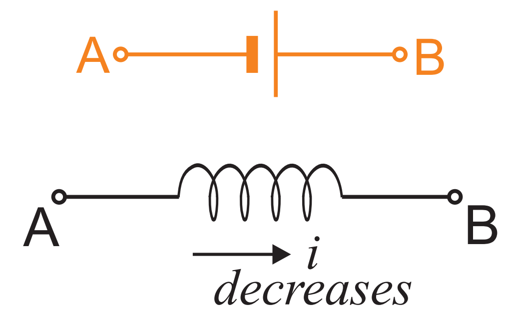
Negative sign comes from Lenz Law

$$\text{Unit of L : Henry (H)} \qquad 1\ \text{H} = 1\frac{\text{Vs}}{\text{A}}$$

- All circuit elements (including resistors) have some inductance
- Commonly used inductors: solenoids and toroids
- circuit symbol 

Example Solenoid





$$\mathcal{E}_L = V_B - V_A = -L\,\frac{di}{dt} < 0 \qquad \therefore \quad V_B < V_A$$

$$\mathcal{E}_L = V_B - V_A = -L\,\frac{di}{dt} > 0 \qquad V_B > V_A$$

Recall **Faraday's Law**

$$\mathcal{E}_L = -N\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(N\Phi_B\right)$$

where Φ_B is magnetic flux ☛ $N\Phi_B$ is flux linkage

$\therefore$ Alternative definition of Inductance

$$-\frac{d}{dt}\left(N\Phi_B\right) = -L\frac{di}{dt} \quad \Rightarrow \quad L = \frac{N\Phi_B}{i}$$

$\therefore$ Inductance is also **flux linkage per unit current**

Calculating Inductance:

① Solenoid 

To first order approximation ➡ $B = \mu_0 n i$

$n = N/\ell$ ☛ number of coils per unit length

Consider a subsection of length l of solenoid

$$\begin{aligned} \text{Flux linkage } &= N\,\Phi_B \\ &= nl \cdot B\,A \quad \text{where } A \text{ is cross-sectional area} \end{aligned}$$

$$\therefore \qquad L = \frac{N\Phi_B}{i} = \mu_0 n^2 l A$$

$$\frac{L}{l} = \mu_0 n^2 A = \textbf{Inductance per unit length}$$

Note

- ❑ $L \propto n^2$
- ❑ Inductance (like capacitance) depends only on geometric factors (not on i)

② Toroid

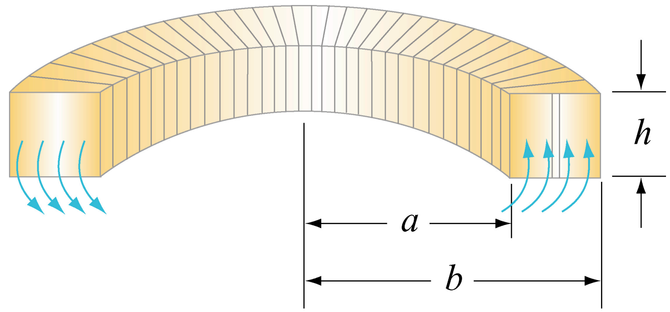
Inside toroid

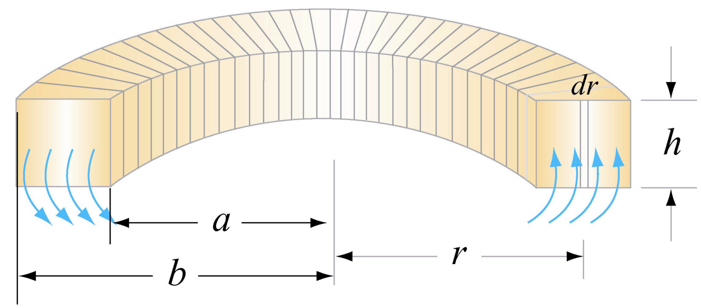
$$B = \frac{\mu_0 i N}{2\pi r}$$

Outside toroid

$$B = 0$$

Recall ☛ B-field lines are concentric circles



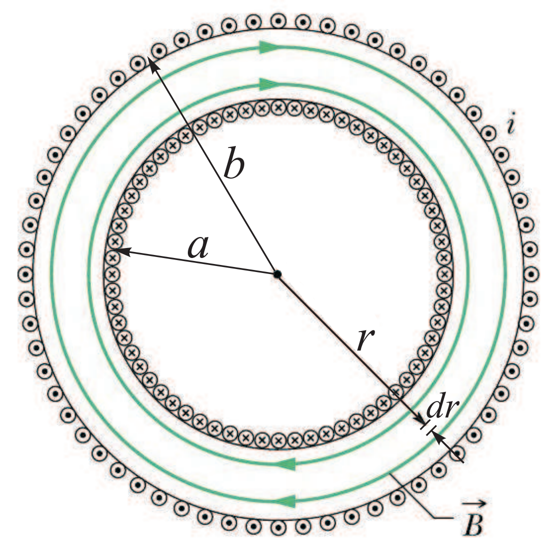


Flux linkage through toroid

$$N\,\Phi_B = N \int \vec{B}\cdot d\vec{a} \left\{ \begin{array}{l} \vec{B} \parallel d\vec{a} \\ da = h\,dr \end{array} \right\rangle \textbf{KEY}$$

$$= \frac{\mu_0 i N^2}{2\pi} \int_a^b \frac{h\,dr}{r}$$

$$= \frac{\mu_0 i N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$



∴ Inductance $L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$

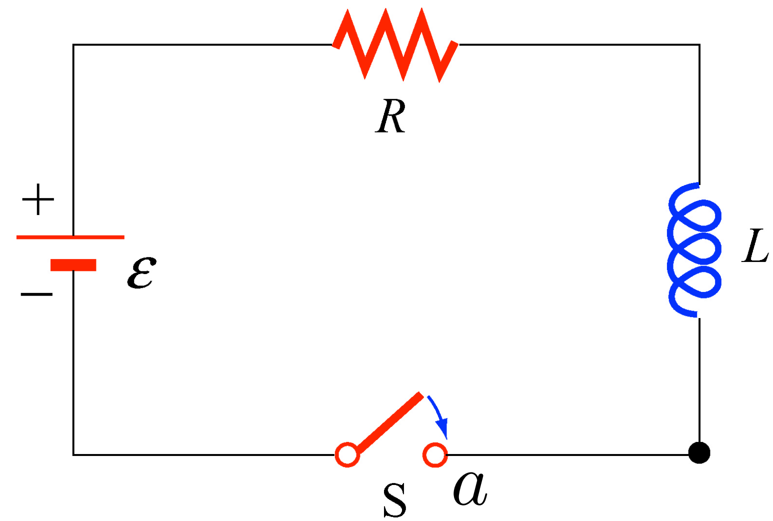
Again $L \propto N^2$

9.2 LR Circuits

(A) **Charging** an inductor

When switch is adjusted to position a

By **loop rule** (clockwise)



$$\mathcal{E}_0 - \Delta V_R + \Delta V_L = 0$$

$$\downarrow \qquad \downarrow$$

$$\mathcal{E}_0 - iR - L\frac{di}{dt} = 0$$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i = \frac{\mathcal{E}_0}{L}$$

First Order Differential Equation

Similar to equation for charging a capacitor!

changing variables

$$x = (\mathcal{E}_0/R) - i \qquad\qquad dx = -di$$

$$x + \frac{L}{R}\frac{dx}{dt} = 0$$

$$\int_{x_o}^{x} \frac{dx'}{x'} = -\frac{R}{L}\int_0^t dt$$

$$\ln(x/x_0) = -Rt/L$$

$$x = x_0 e^{-Rt/L}$$

$$i = 0 \ @ \ t = 0 \Rightarrow x_0 = \mathcal{E}_0/R$$

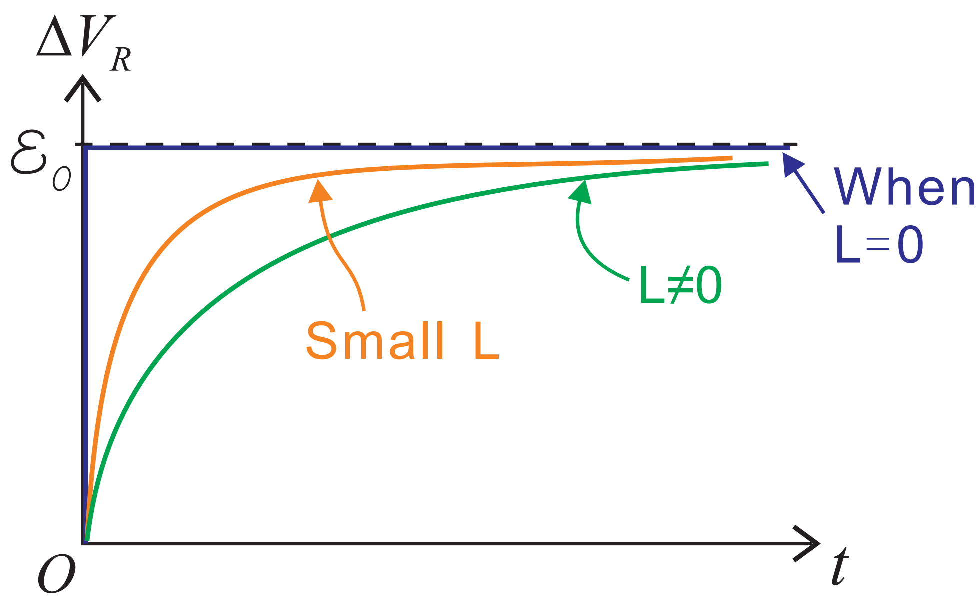
$$\frac{\mathcal{E}_0}{R} - i = \frac{\mathcal{E}_0}{R} e^{-Rt/L}$$

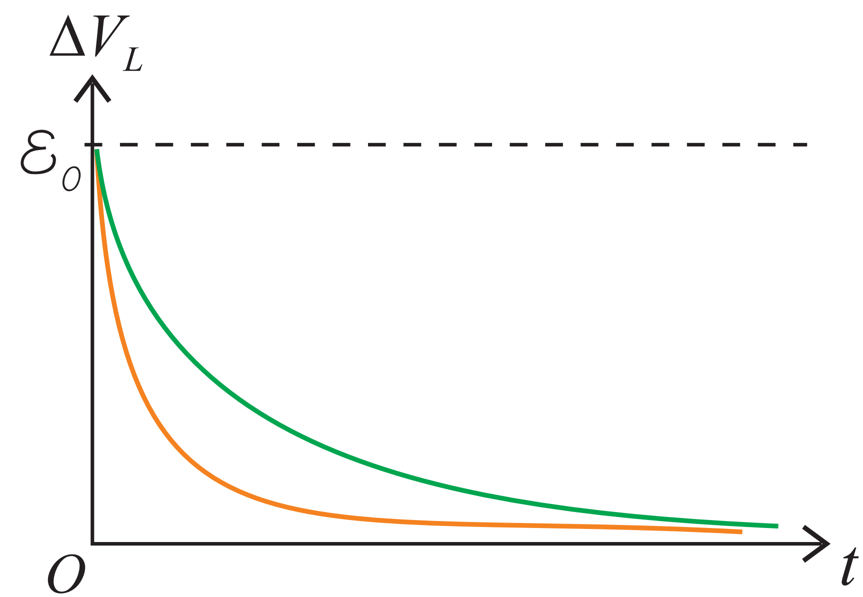
Solution ☛ $$i(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t/\tau_L}\right)$$

$$\tau_L = L/R$$ ☛ **Inductive time constant**

$$|\Delta V_R| = iR = \mathcal{E}_0\left(1 - e^{-t/\tau_L}\right)$$

$$|\Delta V_L| = L\frac{di}{dt} = L \cdot \frac{\mathcal{E}_0}{R} \cdot \frac{1}{\tau_L} \cdot e^{-t/\tau_L} = \mathcal{E}_0 e^{-t/\tau_L}$$





(B) **Discharging** an inductor

When switch is adjusted at position b after inductor has been **charged**

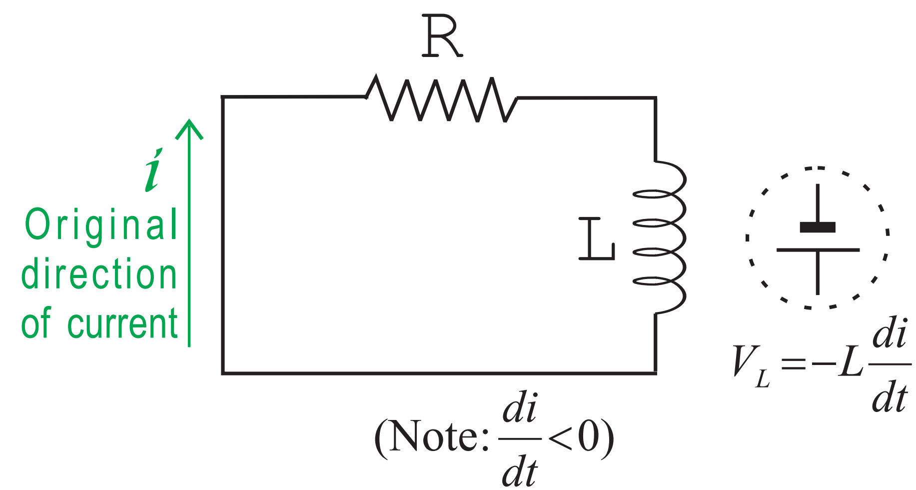
i.e. current $i = \mathcal{E}_0/R$ is flowing in circuit

By **loop rule**

$$\Delta V_L \quad - \quad \Delta V_R \quad = \quad 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$-L\frac{di}{dt} \quad - \quad iR \quad = \quad 0$$

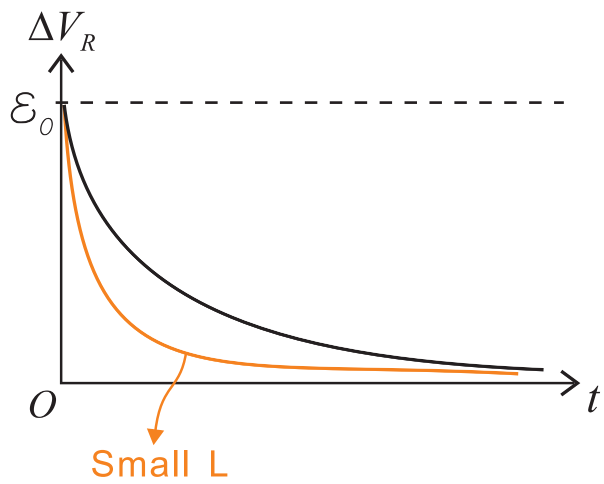


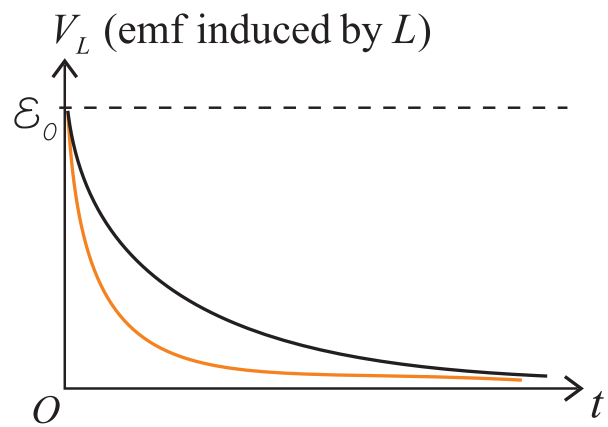
Treat inductor as source of emf

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i = 0$$ Discharging an inductor

$$i(t) = i_0\, e^{-t/\tau_L}$$

where $i_0 = i(t=0) =$ Current when circuit just switch to position b





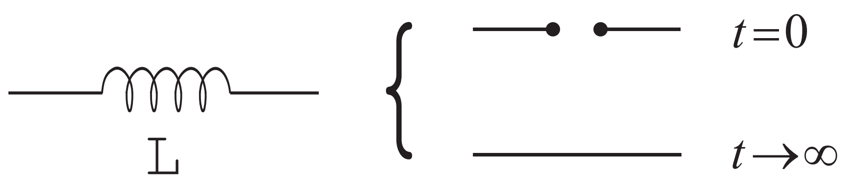
Inductor behaves like a *time-varying battery.*

Summary

During charging of inductor

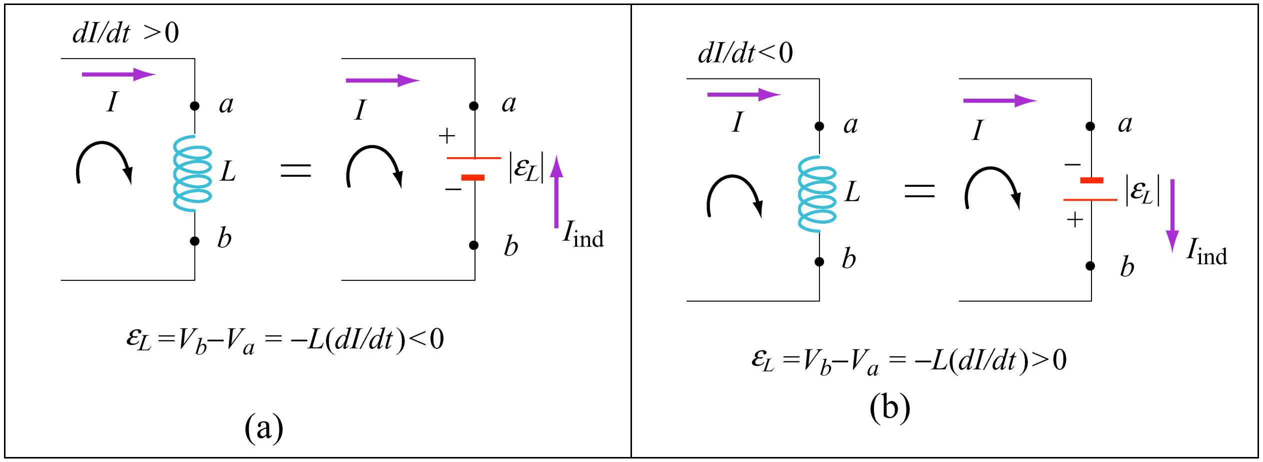
1. At $t = 0$ inductor acts like open circuit when **current flowing is zero**

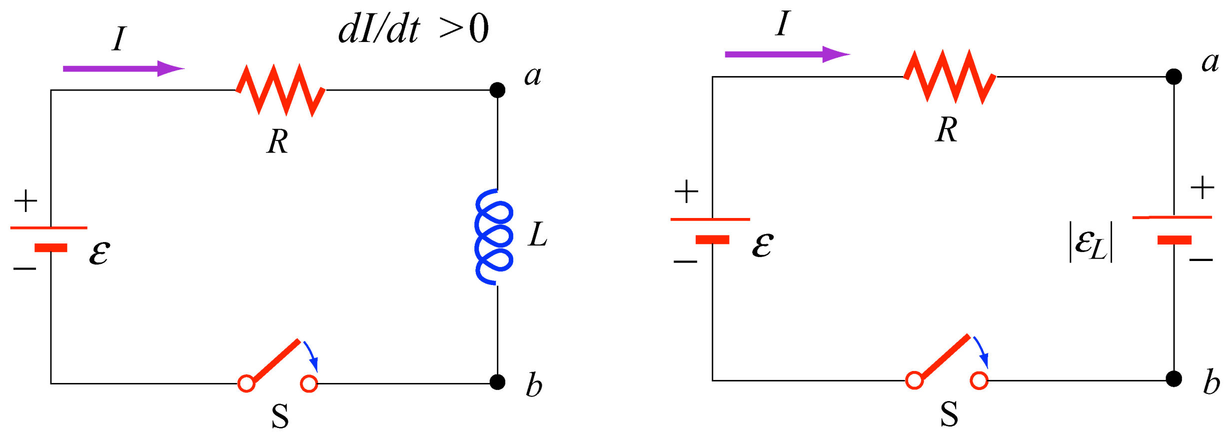
2. At $t \to \infty$ inductor acts like **short circuit** when **current flowing is stabilized at maximum**



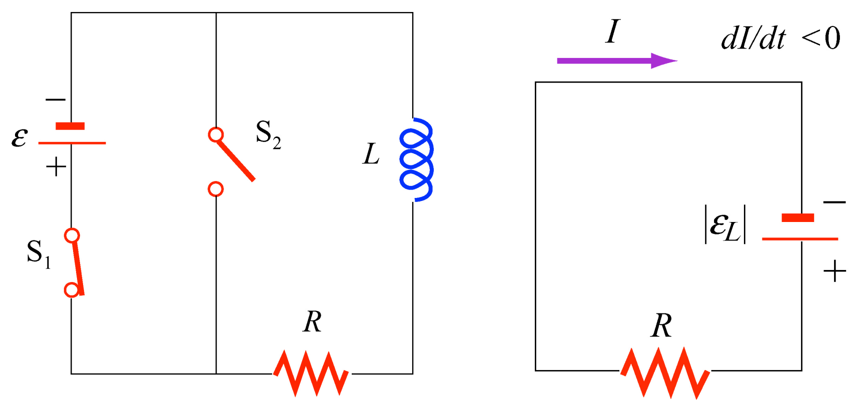
3. Inductors are used everyday in switches for safety concerns

Summary





RL circuit with rising current and equivalent circuit



RL circuit with decaying current and equivalent circuit

9.3 Energy Stored in Inductors

Inductors stored magnetic energy through magnetic field stored in circuit

Recall equation for charging inductors

$$\mathcal{E}_0 - iR - L\frac{di}{dt} = 0$$

Multiply both sides by i

$$\underbrace{\mathcal{E}_0 i}_{\substack{\text{Power input by emf} \\ \text{(Energy supplied} \\ \text{one charge } = q\mathcal{E}_0)}} = \underbrace{i^2 R}_{\substack{\text{Joule's heating} \\ \text{(Power dissipated} \\ \text{by resistor)}}} + \underbrace{Li\frac{di}{dt}}_{\text{Power stored in inductor}}$$

$\therefore$ Power stored in inductor

Integrating both sides and use initial condition

At $t = 0, \quad i(t=0) = U_B(t=0) = 0$

$\therefore$ **Energy stored in inductor** ☛ $U_B = \frac{1}{2}Li^2$

Energy Density Stored in Inductors

Consider an **infinitely long** solenoid of cross-sectional area A

For a portion l of solenoid

$$L = \mu_0 n^2 lA$$

$\therefore$ Energy stored in inductor:

$$U_B = \frac{1}{2} Li^2 = \frac{1}{2}\mu_0 n^2 i^2 \underbrace{lA}_{\text{Volume of solenoid}}$$

$\therefore$ **Energy density** (= Energy stored per unit volume) inside inductor

$$u_B = \frac{U_B}{lA} = \frac{1}{2}\mu_0 n^2 i^2$$

Recall magnetic field inside solenoid

$$B = \mu_0 n i$$

$$\therefore \quad u_B = \frac{B^2}{2\mu_0}$$

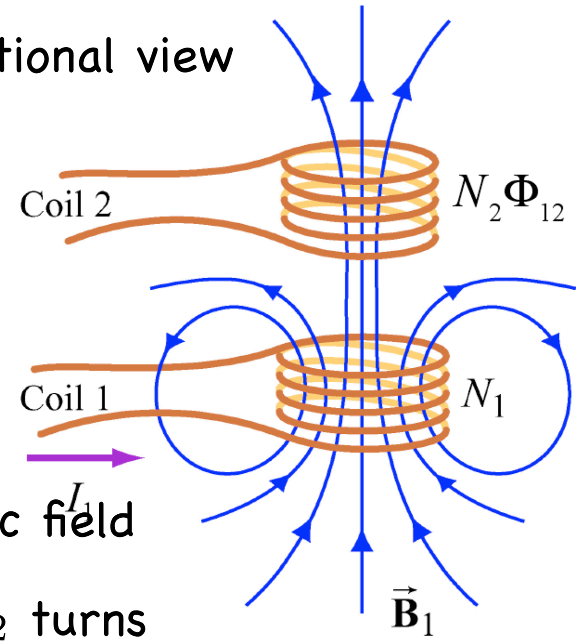
This is a **general result** of **energy stored in a magnetic field**

9.4 Mutual Inductance

Very often the magnetic flux through the area enclosed by a circuit varies with time because of time-varying currents in nearby circuits

mutual inductance depends on interaction of two circuits

Consider two closely wound coils of wire shown in cross-sectional view



Current I_1 in coil 1 which has N_1 turns creates a magnetic field

Some magnetic field lines pass through coil 2 which has N_2 turns

The magnetic flux caused by the current in coil 1 and passing through coil 2 is Φ_{12}

We define the mutual inductance of coil 2 with respect to coil 1

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1}$$

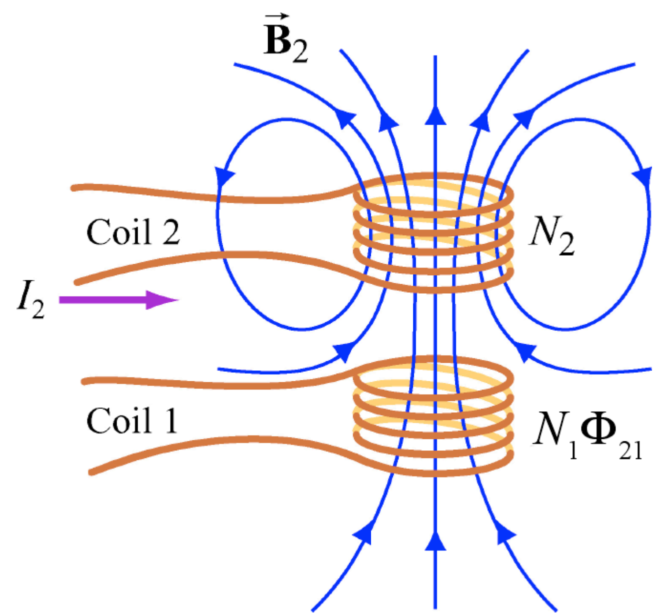
If current I_1 varies with time

we see from Faraday's law that emf induced by coil 1 in coil 2 is

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{12}}{dt} = -N_2 \frac{d}{dt}\left(\frac{M_{12} I_1}{N_2}\right) = -M_{12}\frac{dI_1}{dt}$$

If current I_2 varies with time ☛ emf induced by coil 2 in coil 1 is

$$\mathcal{E}_1 = -M_{21}\frac{dI_2}{dt}$$

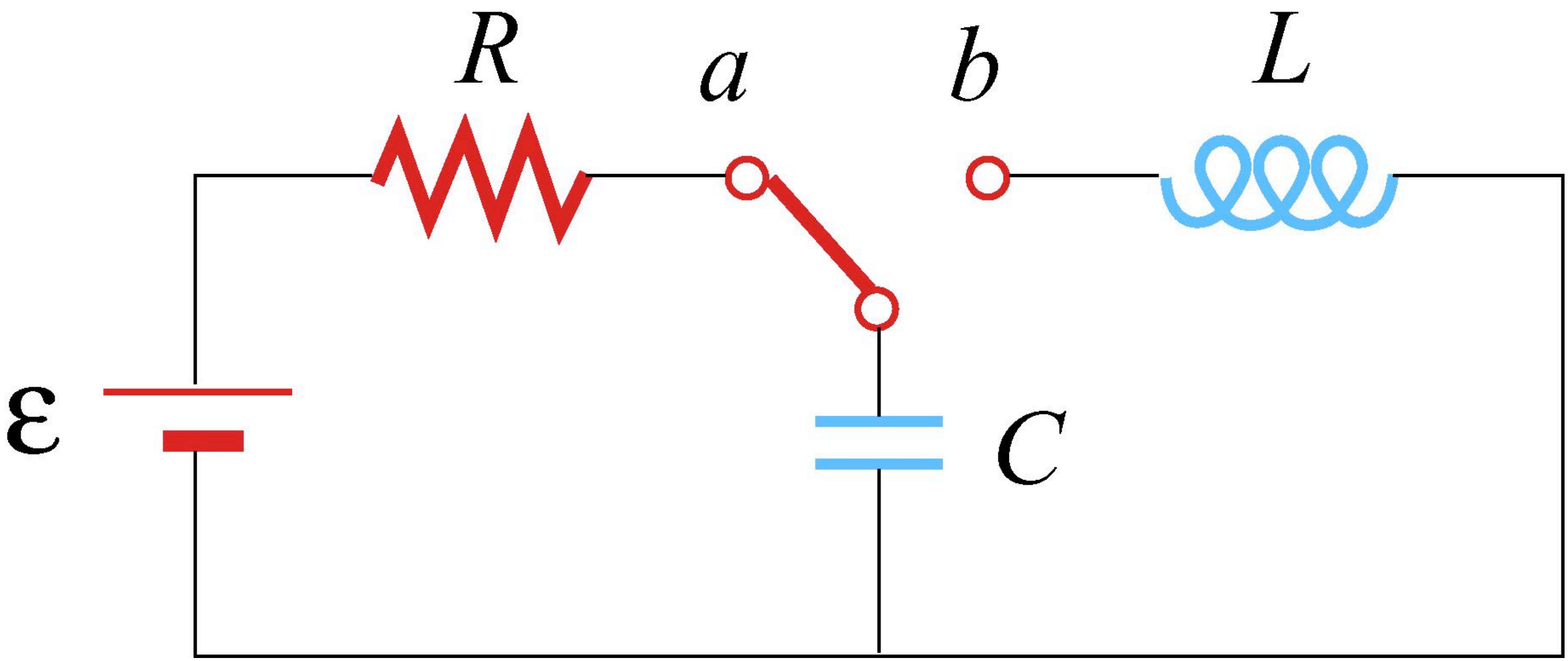


In mutual induction emf induced in one coil

is always proportional to rate at which current in other coil is changing

It is easily seen that $M_{12} = M_{21} = M$

9. 5 LC Circuit (Electromagnetic Oscillator)

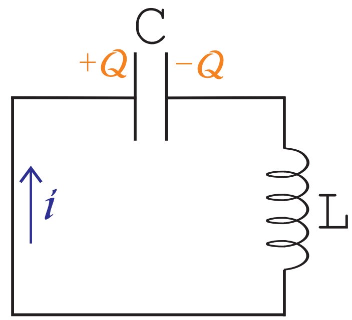


After the capacitor is charged we move the switch to position b

Initial charge on capacitor $= Q$

Initial current $= 0$

No battery



Assume current i to be in direction that **decreases** charge on **positive capacitor plate**

$$\Rightarrow \quad i = \frac{dQ}{dt} \tag{10.1}$$

By **Lenz Law** we also know **poles** of inductor

Loop rule ☛ $\quad V_C + V_L = 0$

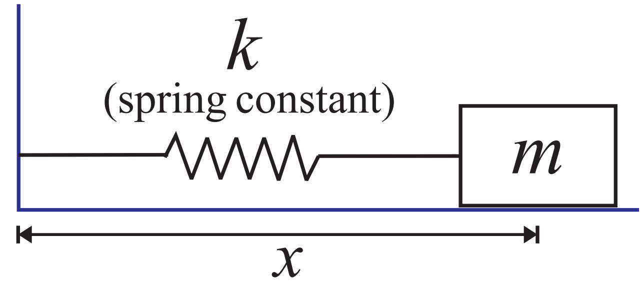
$$-\frac{Q}{C} - L\frac{di}{dt} = 0 \tag{10.2}$$

Combining equations (10.1) and (10.2) we get

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

This is similar to equation of motion of

a **simple harmonic oscillator**



$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Another approach (**conservation of energy**)

Total energy stored in circuit

$$U = U_E + U_B$$

$$U = \frac{Q^2}{2C} + \frac{1}{2}Li^2$$

Since resistance in circuit is zero **no energy** is dissipated in circuit

$\therefore$ Energy contained in circuit is **conserved**

$$\therefore \quad \frac{dU}{dt} = 0$$

$$\Rightarrow \quad \frac{Q}{C}\cdot\frac{dQ}{dt} + Li\frac{di}{dt} = 0 \qquad \left(\because i = \frac{dQ}{dt}\right)$$

$$\Rightarrow \quad L\frac{di}{dt} + \frac{Q}{C} = 0$$

$$\Rightarrow \quad \frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

Solution to this differential equation is in form

$$Q(t) = Q_0 \cos(\omega t + \phi)$$

$$\therefore \quad \frac{dQ}{dt} = -\omega Q_0 \sin(\omega t + \phi)$$

$$\frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi)$$

$$= -\omega^2 Q$$

$$\therefore \quad \frac{d^2Q}{dt^2} + \omega^2 Q = 0$$

$$\therefore \quad \omega^2 = \frac{1}{LC}$$ Angular frequency of LC oscillator

Q_0, ϕ are constants derived from initial conditions

(Two initial conditions, **e.g.** $Q(t=0)$ and $i(t=0) = \left.\dfrac{dQ}{dt}\right|_{t=0}$ are required)

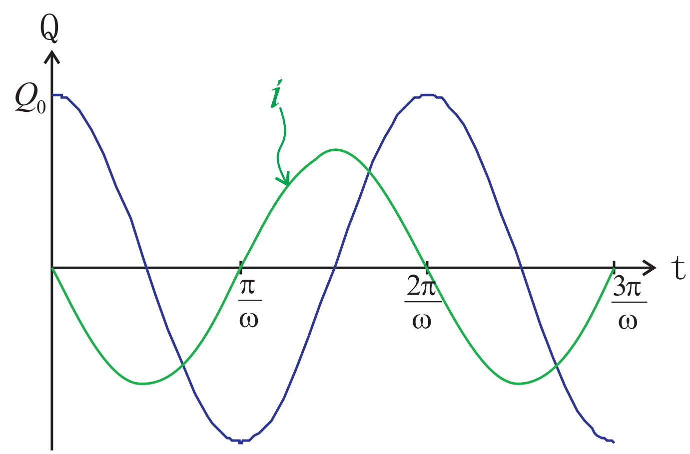
Energy stored in capacitor $= \dfrac{Q^2}{2C} = \dfrac{Q_0^2}{2C}\cos^2(\omega t + \phi)$

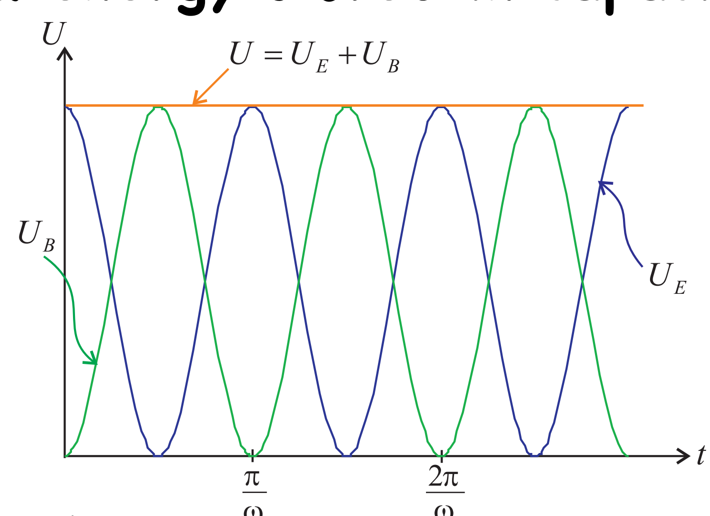
Energy stored in inductance $= \dfrac{1}{2}Li^2 = \dfrac{1}{2}L\omega^2 Q_0^2 \sin^2(\omega t + \phi)$

(Since $L\omega^2 = \dfrac{1}{C}$) $= \dfrac{Q_0^2}{2C}\sin^2(\omega t + \phi)$

$\therefore$ Total energy stored $= \dfrac{Q_0^2}{2C}$

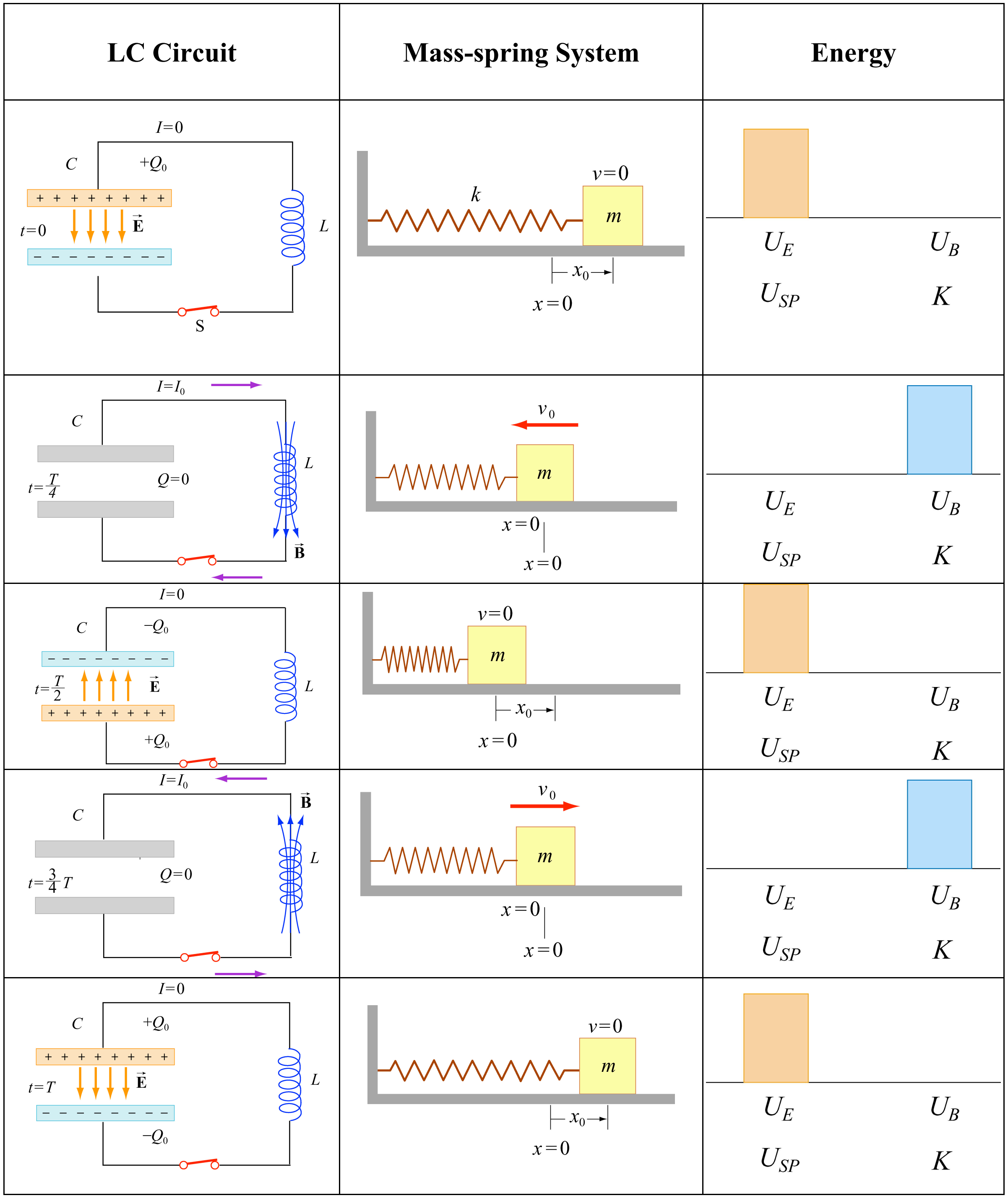
$=$ **Initial energy stored in capacitor**





Assume $\phi = 0$ here

Energy oscillations in LC system and mass-spring system



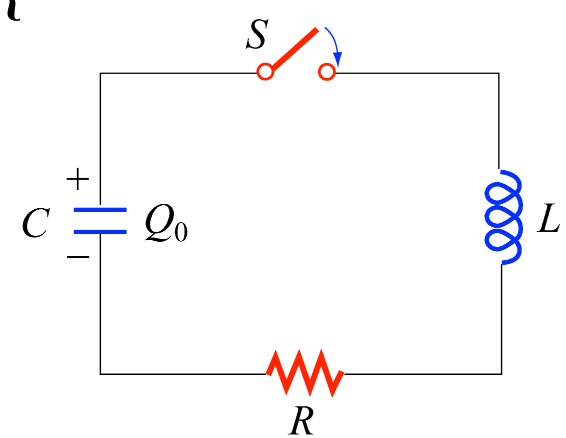
9. 6 RLC Circuit (Damped Oscillator)

In real life circuit ☛ there's **always** resistance

energy stored in LC oscillator is **NOT** conserved

and

$$\frac{dU}{dt} = \text{Power dissipated in resistor} = -i^2 R$$



Negative sign shows that energy U is decreasing

Joule's heating

$$\therefore \quad Li\frac{di}{dt} + \frac{Q}{C} \cdot \overbrace{\frac{dQ}{dt}}^{i} = -i^2 R$$

$$\Rightarrow \quad \frac{d^2 Q}{dt^2} + \frac{R}{L} \cdot \frac{dQ}{dt} + \frac{1}{LC} Q = 0$$

This is similar to equation of motion of a **damped harmonic oscillator** (**e.g.** if a mass-spring system faces a frictional force $\vec{F} = -b\vec{v}$)

Solution to equation is of form

$$Q(t) = Q_0 \underbrace{e^{-\frac{R}{2L}t}}_{\text{exponential decay term}} \underbrace{\cos(\omega' t + \phi)}_{\text{oscillating term}}$$

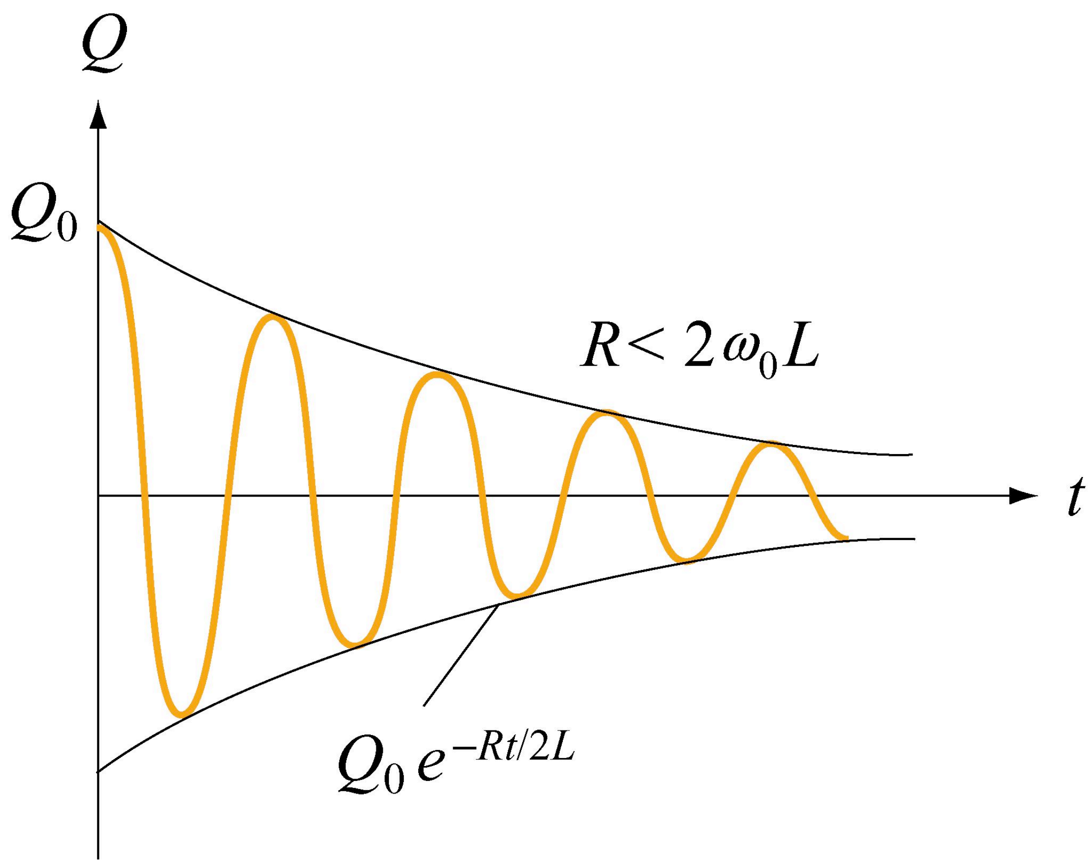
$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$$

$$\gamma = \frac{R}{2L}$$ damping factor

There are three possible scenarios depending on the relative values of γ and ω_0

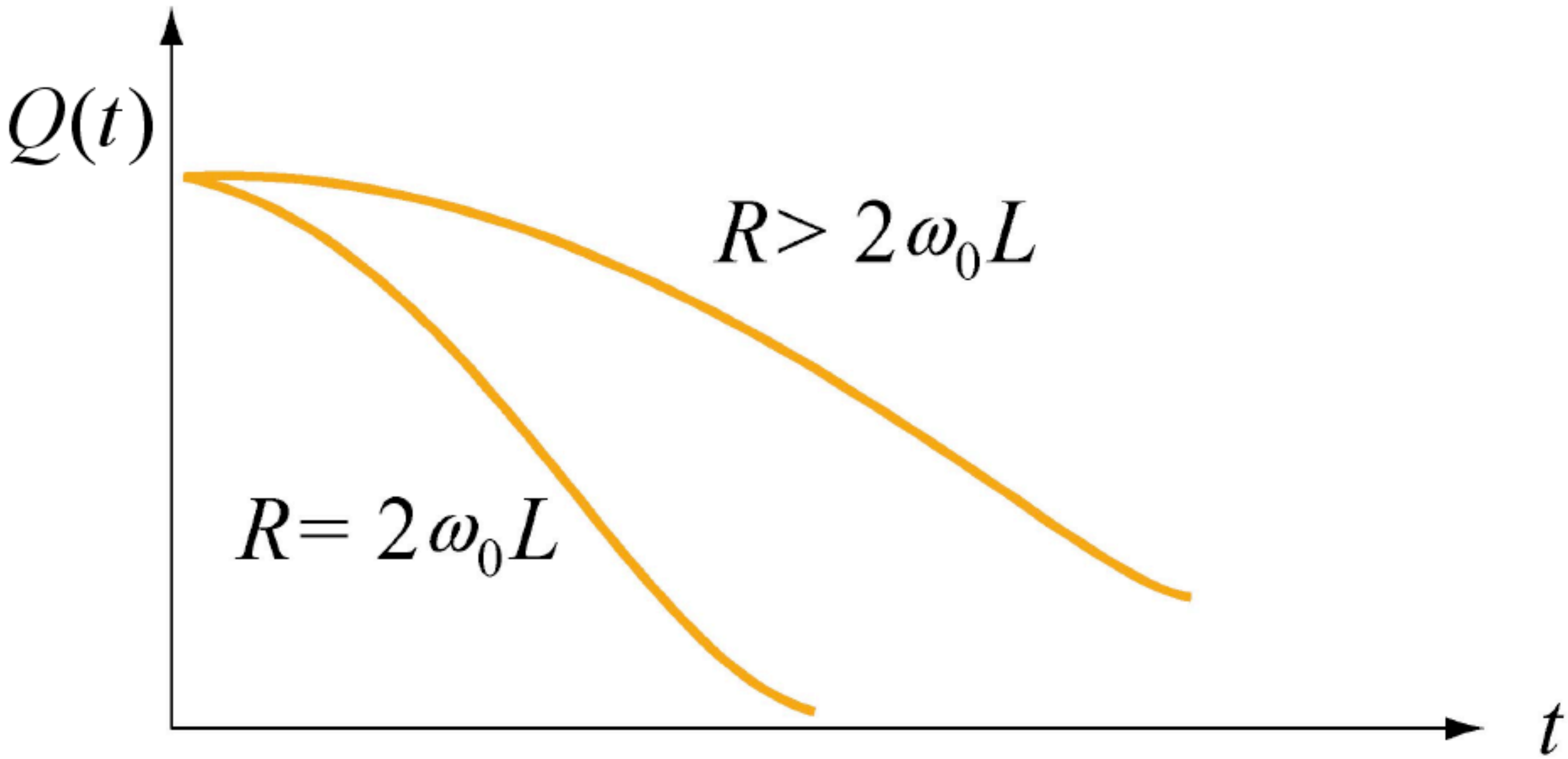
Case I: Underdamping $\omega_0 > \gamma$



Underdamped oscillator always oscillates

at a **lower** frequency than **natural frequency** of oscillator

Case II: Overdamping $\omega_0 < \gamma$



Case III: Critical damping $\omega_0 = \gamma$

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