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Problems set # 5 **Physics 169** March 17, 2015

1. *(i)* Three capacitors are connected to a 12.0 V battery as shown in Fig. 1 Their capacitances are $C_1 = 3.00\ \mu$F, $C_2 = 4.00\ \mu$F, and $C_3 = 2\mu$F. Find the equivalent capacitance of this set of capacitors. *(ii)* Find the charge on and the potential difference across each.

Solution *(i)* Using the rules for combining capacitors in series and in parallel, the circuit is reduced in steps as shown in Fig. 2. The equivalent capacitor is hown to be a 2.00 μF capacitor. *(ii)* From Fig. 2 (right panel) it follows that $Q_{ac} = C_{ac}(\Delta V)_{ac} = 2.00\ \mu\text{F}\ 12.0V = 24.0\ \mu$C. From Fig. 2 (middle panel) it follows that $Q_{ab} = Q_{bc} = Q_{ac} = 24\ \mu$C and so the charge on the 3.00μF capacitor is $Q_3 = 24.0\ \mu$C. Continuing to use Fig. 2 (middle panel) it follows that $(\Delta V)_{ab} = \frac{Q_{ab}}{C_{ab}} = \frac{24.0\ \mu\text{C}}{6.00\ \mu\text{F}} = 4.00$ V, and $(\Delta V)_3 = (\Delta V)_{bc} = \frac{Q_{bc}}{C_{bc}} = \frac{24.0\ \mu\text{C}}{3.00\ \mu\text{F}} = 8.00$ V. Finally, from Fig. 2 (left panel) it follows that $(\Delta V)_4 = (\Delta V)_2 = (\Delta V)_{ab} = 4.00$ V, $Q_4 = C_4(\Delta V)_4 = 4.00\ \mu\text{F}\ 4.00\ \text{V} = 16.0\ \mu$C, and $Q_2 = C_2(\Delta V)_2 = 2.00\ \mu\text{F}4.00\ \text{V} = 8.00\ \mu$C.

2. A dielectric rectangular slab has length s, width w, thickness d, and dielectric constant κ. The slab is inserted on the right hand side of a parallel-plate capacitor consisting of two conducting plates of width w, length L, and thickness d. The left hand side of the capacitor of length $L - s$ is empty, see Fig. 3. The capacitor is charged up such that the left hand side has surface charge densities $\pm\sigma_L$ on the facing surfaces of the top and bottom plates respectively and the right hand side has surface charge densities $\pm\sigma_R$ on the facing surfaces of the top and bottom plates respectively. The total charge on the entire top and bottom plates is $+Q$ and $-Q$ respectively. The charging battery is then removed from the circuit. Neglect all edge effects. *(i)* Find an expression for the magnitude of the electric field E_L on the left hand side in terms of σ_L, σ_R, κ, s, w, L, ϵ_0, and d as needed. *(ii)* Find an expression for the magnitude of the electric field E_R on the right hand side in terms of σ_L, σ_R, κ, s, w, L, ϵ_0, and d as needed. *(iii)* Find an expression that relates the surface charge densities σ_L and σ_R in terms of κ, s, w, L, ϵ_0, and d as needed. *(iv)* What is the total charge $+Q$ on the entire top plate? Express your answer in terms of σ_L, σ_R, κ, s, w, L, ϵ_0, and d as needed. *(v)* What is the capacitance of this system? Express your answer in terms of κ, s, w, L, ϵ_0, and d as needed. *(vi)* Suppose the dielectric is removed. What is the change in the stored potential energy of the capacitor? Express your answer in terms of Q, κ, s, w, L, ϵ_0, and d as needed.

Solution: *(i)* Using Gauss' law $E_L = \frac{\sigma_L}{\epsilon_0}$. *(ii)* Using Gauss law for dielectrics $E_R = \frac{\sigma_R}{\kappa\epsilon_0}$. *(iii)* The potential difference on the left side is $E_L d = \frac{\sigma_L d}{\epsilon_0}$. On the right hand side it is $E_R d = \frac{\sigma_R d}{\kappa\epsilon_0}$. Since these must be equal we must have $\sigma_R/\kappa = \sigma_L$. *(iv)* $Q = \sigma_L(L - s)w + \sigma_R s w$. *(v)* The potential difference is $E_L d = \frac{\sigma_L d}{\epsilon_0}$, so the capacitance is $C = \frac{Q}{|\Delta V|} = \frac{\sigma_L(L-s)w+\sigma_R s w}{\sigma_L d/\epsilon_0} = \frac{\epsilon_0 w}{d}\left[(L - s) + \frac{\sigma_R}{\sigma_L}s\right] = \frac{\epsilon_0 w}{d}\left[(L - s) + \kappa s\right]$. *(vi)* Since the battery has been removed, the charge on the capacitor does not change when we do this, and the change in the energy stored is $= \frac{1}{2}\frac{Q^2}{C_0} - \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2}\left(\frac{1}{C_0} - \frac{1}{C}\right) = \frac{Q^2}{2}\left\{\frac{d}{\epsilon_0 L w} - \frac{d}{\epsilon_0 w[(L-s)+\kappa s]}\right\} = \frac{Q^2 d}{2\epsilon_0 w}\left[\frac{1}{L} - \frac{1}{(L-s)+\kappa s}\right] = \frac{Q^2 d}{2\epsilon_0 w}\left\{\frac{(L-s)+\kappa s - L}{L[(L-s)+\kappa s]}\right\} = \frac{Q^2 d}{2\epsilon_0 w}\left\{\frac{(\kappa-1)s}{L[(L-s)+\kappa s]}\right\}$.

3. *(i)* Consider a plane-parallel capacitor completely filled with a dielectric material of dielectric constant κ. What is the capacitance of this system? *(ii)* A parallel-plate capacitor is constructed by filling the space between two square plates with blocks of three dielectric materials, as in Fig. 4. You may assume that $l \gg d$. Find an expression for the capacitance of the device in terms of the plate area A, d, κ_1, κ_2, and κ_3.

Solution: *(i)* The capacitance is $C = \frac{\kappa \epsilon_0 A}{d} = \kappa C_0$. *(ii)* The capacitor can be regarded as being consisted of three capacitors, $C_1 = \frac{\kappa_1 \epsilon_0 A/2}{d}$, $C_2 = \frac{\kappa_2 \epsilon_0 A/2}{d/2}$, and $C_3 = \frac{\kappa_3 \epsilon_0 A/2}{d/2}$, with C_2 and C_3 connected in series, and the combination connected in parallel with C_1. Thus, the equivalent capacitance is $C = C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} = C_1 + \frac{C_2 C_3}{C_2 + C_3} = \frac{\kappa_1 \epsilon_0 A/2}{d} + \frac{\epsilon_0 A}{d}\left(\frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\right) = \frac{\epsilon_0 A}{d}\left(\frac{\kappa_1}{2} + \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\right)$.

4. A model of a red blood cell portrays the cell as a spherical capacitor – a positively charged liquid sphere of surface area A, separated by a membrane of thickness t from the surrounding negatively charged fluid. Tiny electrodes introduced into the interior of the cell show a potential difference of 100 mV across the membrane. The membranes thickness is estimated to be 100 nm and its dielectric constant to be 5.00. *(i)* If an average red blood cell has a mass of 1.00×10^{-12} kg, estimate the volume of the cell and thus find its surface area. The density of blood is $1,100\ \mathrm{kg/m^3}$. *(ii)* Estimate the capacitance of the cell. *(iii)* Calculate the charge on the surface of the membrane. How many electronic charges does this represent?

Solution *(i)* The volume is $V = \frac{m}{\rho} = \frac{1.00 \times 10^{-12}\ \mathrm{kg}}{1{,}100\ \mathrm{kg/m^3}} = 9.09 \times 10^{-16}\ \mathrm{m^3}$. Since $V = \frac{4}{3}\pi r^3$, the inner radius of the capacitor is $r = \left(\frac{3V}{4\pi}\right)^{1/3} = 6.01 \times 10^{-6}$ m and the surface area is $A = 4\pi r^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3} = 4\pi\left(\frac{3}{4\pi} 9.09 \times 10^{-16}\ \mathrm{m^3}\right)^{2/3} = 4.54 \times 10^{-10}\ \mathrm{m^2}$. *(ii)* The outer radius of the capacitor is $R = r + t = 6.11 \times 10^{-6}$ m, where $t = 100$ nm is the thickness of the membrane. The capacitance is $C = 4\pi\kappa\epsilon_0 \frac{Rr}{R-r} = 2.04 \times 10^{-13}$ F. *(iii)* $Q = C\,\Delta V = 2.04 \times 10^{-13}\ \mathrm{F}\ 100 \times 10^{-3}\ \mathrm{V} = 2.01 \times 10^{-14}$ C, and the number of electronic charges is $n = \frac{Q}{e} = \frac{2.01 \times 10^{-14}\ \mathrm{C}}{1.60 \times 10^{-19}\ \mathrm{C}} = 1.27 \times 10^5$.

5. A capacitor consists of two concentric spherical shells. The outer radius of the inner shell is $a = 0.1$ m and the inner radius of the outer shell is $b = 0.2$ m. *(i)* What is the capacitance C of this capacitor? *(ii)* Suppose the maximum possible electric field at the outer surface of the inner shell before the air starts to ionize is $E_{\max}(a) = 3.0 \times 10^6\ \mathrm{V \cdot m^{-1}}$. What is the maximum possible charge on the inner capacitor? *(iii)* What is the maximum amount of energy stored in this capacitor? *(iv)* What is the potential difference between the shells when $E(a) = 3.0 \times 10^6\ \mathrm{V \cdot m^{-1}}$?

Solution: *(i)* The shells have spherical symmetry so we need to use spherical Gaussian surfaces. Space is divided into three regions (I) outside $r \geq b$, (II) in between $a < r < b$ and (III) inside $r \leq a$. In each region the electric field is purely radial (that is $\vec{E} = E\hat{r}$). In regions I and III these Gaussian surfaces contain a total charge of zero, so the electric fields in these regions must be zero as well. In region II, we choose the Gaussian sphere of radius r shown in Fig. 5. The electric flux on the surface is $\oiint \vec{E} \cdot d\vec{A} = EA = E \cdot 4\pi r^2$. The enclosed charge is $Q_{\mathrm{enc}} = +Q$, and the electric field is everywhere perpendicular to the surface. Thus Gauss law becomes $E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$.

That is, the electric field is exactly the same as that for a point charge. Summarizing:

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for } a < r < b \\ \vec{0} & \text{elsewhere} \end{cases} .$$

We know the positively charged inner sheet is at a higher potential so we shall calculate $\Delta V = V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{s} = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}dr = \left.\frac{Q}{4\pi\epsilon_0 r}\right|_b^a = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) > 0$, which is positive as we expect. We can now calculate the capacitance using the definition $C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\epsilon_0 ab}{b-a} = \frac{0.1\text{ m }0.2\text{ m}}{8.99\times10^9\text{ Nm}^2/\text{C}^2\ 0.1\text{ m}} = 2.2\times10^{-11}$ F. Note that the units of capacitance are ϵ_0 times an area ab divided by a length $b-a$, exactly the same units as the formula for a parallel-plate capacitor $C = \epsilon_0 A/d$. Also note that if the radii b and a are very close together, the spherical capacitor begins to look very much like two parallel plates separated by a distance $d = b - a$ and area $A \approx 4\pi\left(\frac{a+b}{2}\right)^2 \approx 4\pi\left(\frac{a+a}{2}\right)^2 = 4\pi a^2 \approx 4\pi ab$. So, in this limit, the spherical formula is the same at the plate one $C = \lim_{b\to a}\frac{4\pi\epsilon_0 ab}{b-a} \approx \frac{\epsilon_0 4\pi a^2}{d} = \frac{\epsilon_0 A}{d}$. *(ii)* The electric field is $E(a) = \frac{Q}{4\epsilon_0 a^2}$. Therefore the maximum charge is $Q_{\text{max}} = 4\pi\epsilon_0 E_{\text{max}}(a)a^2 = \frac{3.0\times10^6\text{ V}\cdot\text{m}^{-1}\ (0.1\text{ m})^2}{8.99\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2}$. *(iii)* The energy stored is $U_{\text{max}} = \frac{Q_{\text{max}}^2}{2C} = \frac{(3.3\times10^{-6}\text{ C})^2}{2\cdot2.2\times10^{-11}\text{ F}} = 2.5\times10^{-1}$ J. *(iv)* We can find the potential difference two different ways. Using the definition of capacitance we have that $|\Delta V| = \frac{Q}{C} = \frac{4\pi\epsilon_0 E(a)a^2(b-a)}{4\pi\epsilon_0 ab} = \frac{E(a)a(b-a)}{b} = \frac{3.0\times10^6\text{ V}\cdot\text{m}^{-1}(0.1\text{ m})^2}{0.2\text{ m}} = 1.5\times10^5$ V. We already calculated the potential difference in part *(i)*: $\Delta V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$. Recall that $E(a) = \frac{Q}{4\pi\epsilon_0 a^2}$ or $\frac{Q}{4\pi\epsilon_0} = E(a)a^2$ Substitute this into our expression for potential difference yielding $\Delta V = E(a)a^2\left(\frac{1}{a}-\frac{1}{b}\right) = E(a)a^2\frac{b-a}{ab} = E(a)a\frac{b-a}{b}$ in agreement with our result above.

6. A parallel plate capacitor has capacitance C. It is connected to a battery until is fully charged, and then disconnected. The plates are then pulled apart an extra distance d, during which the measured potential difference between them changed by a factor of 4. What is the volume of the dielectric necessary to fill the region between the plates? (Make sure that you give your answer only in terms of variables defined in the statement of this problem, fundamental constants and numbers).

Solution How in the world do we know the volume? We must be able to figure out the cross-sectional area and the distance between the plates. The first relationship we have is from knowing the capacitance: $C = \frac{\epsilon_0 A}{x}$ where x is the original distance between the plates. Make sure you dont use the more typical variable d here because that is used for the distance the plates are pulled apart. Next, the original voltage $V_0 = Ex$, which increases by a factor of 4 when the plates are moved apart by a distance d, that is, $4V_0 = E(x+d)$. From these two equations we can solve for x: $4V_0 = 4Ex = E(x+d) \Rightarrow x = d/3$. Now, we can use the capacitance to get the area, and multiply that by the distance between the plates (now $x+d$) to get the volume, i.e., volume $= A(x+d) = \frac{xC}{\epsilon_0}(x+d) = \frac{dC}{3\epsilon_0}\left(\frac{d}{3}+d\right) = \frac{4d^2C}{9\epsilon_0}$.

7. Consider two nested cylindrical conductors of height h and radii a and b respectively. A charge $+Q$ is evenly distributed on the outer surface of the pail (the inner cylinder), $-Q$ on the inner surface of the shield (the outer cylinder). See Fig. 6. You may ignore edge effects. *(i)* Calculate the electric field between the two cylinders ($a < r < b$). *(ii)* Calculate the potential difference

between the two cylinders. *(iii)* Calculate the capacitance of this system, $C = Q/\Delta V$ *(iv)* Numerically evaluate the capacitance, given: $h \simeq 15$ cm, $a \simeq 4.75$ cm and $b \simeq 7.25$ cm. *(v)* Find the electric field energy density at any point between the conducting cylinders. How much energy resides in a cylindrical shell between the conductors of radius r (with $a < r < b$), height h, thickness dr, and volume $2\pi r h dr$? Integrate your expression to find the total energy stored in the capacitor and compare your result with that obtained using $U_E = \frac{1}{2}C(\Delta V)^2$.

Solution: *(i)* For this we use Gauss law, with a Gaussian cylinder of radius r, height l: $\oiint \vec{E}\cdot d\vec{A} = 2\pi r l E = \frac{Q_{\rm inside}}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{Q}{h}l \Rightarrow E(r) = \frac{Q}{2\pi r \epsilon_0 h}$ with $a < r < b$. *(ii)* The potential difference between the outer shell and the inner cylinder is $\Delta V = V(a) - V(b) = -\int_b^a \frac{Q}{2\pi r' \epsilon_0 h} dr' = -\frac{Q}{2\pi\epsilon_0 h}\ln r'\Big|_b^a = \frac{Q}{2\pi\epsilon_0 h}\ln\left(\frac{b}{a}\right)$. *(iii)* $C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Q}{2\pi\epsilon_0 h}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon_0 h}{\ln\left(\frac{b}{a}\right)}$. *(iv)* $C = \frac{2\pi\epsilon_0 h}{\ln(b/a)} = 15\ \text{cm}\cdot 2\pi\cdot 8.85\times 10^{-14}\ \text{cm}^{-1}\cdot \text{F}\,\frac{1}{\ln(7.25\ \text{cm}/4.75\ \text{cm})} \simeq 20$ pF. *(v)* The total energy density stored in the capacitor is $u = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0\left(\frac{Q}{2\pi r\epsilon_0 h}\right)^2$. Then $dU = u dV = \frac{1}{2}\epsilon_0\left(\frac{Q}{2\pi\epsilon_0 r h}\right)^2 2\pi r h dr = \frac{Q^2}{4\pi\epsilon_0 h}\frac{dr}{r}$. Integrating we find that $U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\epsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\epsilon_0 h}\ln(b/a)$. From part *(iii)* $C = 2\pi\epsilon_0 h/\ln(b/a)$, therefore $U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\epsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\epsilon_0 h}\ln(b/a) = \frac{Q^2}{2C} = \frac{1}{2}C(\Delta V)^2$, which agrees with that obtained above.

8. A capacitor is made of three sets of parallel plates of area A, with the two outer plates on the left and the right connected together by a conducting wire as shown in Fig. 7. The outer plates are separated by a distance d. The distance from the middle plate to the left plate is z. The distance from the inner plate to the right plate is $d - z$. You may assume all three plates are very thin compared to the distances d and z . Neglect edge effects. *(i)* The positive terminal of a battery is connected to the outer plates. The negative terminal is connected to the middle plate. The potential difference between the outer plates and inner plate is $\Delta V = V(z = 0) - V(z)$. Find the capacitance of this system. *(ii)* Find the total energy stored in this system.

Solution When the battery is connected positive charges Q_L and Q_R appear on the outer plates (on the inner facing surfaces) and negative charges $-Q_L$ and $-Q_R$ respectively appear on each side of the inner plates. The plates on the left in Fig. 8 act as a capacitor arranged as show in Fig. 9. This is a parallel plate capacitor with capacitance $C_L = Q_L/\Delta V$. The plates on the right also act as a capacitor with capacitance $C_R = Q_R/\Delta V$. All the plates have the same area A so neglecting edge effects we can use Gauss law to show that the magnitude of the electric field on the left is $E_L = Q_L/\epsilon_0 A$ for parallel plates on the left. Because the electric field is uniform the potential difference is $\Delta V = E_L z = \frac{Q_L z}{\epsilon_0 A}$. Therefore the positive charge on the left satisfies $Q_L = \frac{\Delta V\,\epsilon_0 A}{z}$. A similar argument applied to the plates on the right show that $\Delta V = E_R(d-z) = \frac{Q_R}{\epsilon_0 A}(d-z)$. Hence the positive charge on the right satisfies $Q_R = \frac{\Delta V \epsilon_0 A}{d-z}$. The total positive charge at the higher potential is the sum $Q_L + Q_R$ capacitance of the capacitor is $C = \frac{Q_L+Q_R}{\Delta V}$. We can now substitute the equations above and solve for the capacitance $C = \frac{\Delta V\epsilon_0 A}{\Delta V z} + \frac{\Delta V\epsilon_0 A}{\Delta V(d-z)} = \epsilon_0 A\left(\frac{1}{z} + \frac{1}{d-z}\right) = \frac{\epsilon_0 A d}{z(d-z)}$. Note: Because the outer plates are at the same potential you can think of these capacitors as connected in parallel. Therefore the equivalent capacitance is $C = C_L + C_R = \frac{Q_L}{\Delta V} + \frac{Q_R}{\Delta V}$ agreeing with our result above. *(ii)* The stored energy is just $U = \frac{1}{2}C(\Delta V)^2 = \frac{\epsilon_0 A d(\Delta V)^2}{2z(d-z)}$.

9. Two flat, square metal plates have sides of length L, and thickness $s/2$, are arranged parallel to each other with a separation of s, where $s \ll L$ so you may ignore fringing fields. A charge Q is moved from the upper plate to the lower plate. Now a force is applied to a third uncharged conducting plate of the same thickness $s/2$ so that it lies between the other two plates to a depth x, maintaining the same spacing $s/4$ between its surface and the surfaces of the other two. The configuration is shown in Fig. 10. *(i)* What is the capacitance of this system? *(ii)* How much energy is stored in the electric field? *(iii)* If the middle plate is released, it starts to move. Will it move to the right or left? [*Hint*: If the middle plate moves to the left by a small positive amount Δx, the change in potential energy is approximately $\Delta U = (dU/dx)\Delta x$. Will the stored potential energy increase or decrease? *(iv)*) For a small displacement Δx in the direction you determined in part *(iii)*, find the horizontal force exerted by the charge distribution on the outer plates acting on the charges on the middle plate that cause it to move.

Solution: *(i)* Divide the plates into left and right sides. The area on the left is $A_L = L(L-x)$ and the area on the right is $A_R = Lx$. The charge densities on the two sides are shown in Fig.11. The charge on the left is$Q_L = \sigma_L L(L-x)$. The charge on the right is $Q_R = \sigma_R(Lx)$. The electric field on the two sides is shown in Fig. 11. We can calculate the electric field on both sides using Gauss law (neglecting edge effects). We find $E_L = \sigma/\epsilon_0$, $E_R = \sigma/\epsilon_0$. Because the electric field is uniform on both sides (neglecting edge effects) $\Delta V_L = E_L s = \sigma_L s/\epsilon_0$. Note that on the right side the electric field is zero in the middle conductor so $\Delta V_R = \frac{\sigma_R s/4}{\epsilon_0} + \frac{\sigma_R s/4}{\epsilon_0}\frac{\sigma_R s}{2\epsilon_0}$. The potential difference on the two sides are the same because the upper and lower plates are held at the same potential difference $\Delta V \equiv \Delta V_L = \Delta V_R$. Therefore we can solve for a relationship between the charge densities on the two sides by setting the above equations equal to each other $\frac{\sigma_L s}{\epsilon_0} = \frac{\sigma_R s}{2\epsilon_0}$; hence $\sigma_L = \frac{\sigma_R}{2}$. The capacitance of the system is the total charge divided by the potential difference $C = \frac{Q_L + Q_R}{\Delta V}$. Using our results for the charges and the potential difference we have that $C = \frac{\sigma_L L(L-x) + \sigma_R Lx}{\sigma_L s/\epsilon_0}$. We can now substitute the value of σ_L and find that the capacitance is $C = \frac{\sigma L(L-x)/2 + \sigma_R Lx}{\sigma_R s/(2\epsilon_0)} = \frac{\epsilon_) [L(L-x)+2Lx]}{s} = \frac{\epsilon_0 L(L+x)}{s}$. *(ii)* The energy stored in the capacitor is then $U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\epsilon_0 L(L+x)}$. *(iii)* If the middle plate moves a positive distance Δx to the left resulting in a larger value of x, then the stored potential energy changes by an amount $\Delta U = \frac{dU}{dx}\Delta x = -\frac{Q^2 s}{2\epsilon_0 L(L+x)^2}\Delta x$. This change is negative hence the middle plate will move to the left decreasing the stored potential energy of the system. Because the change in potential energy is negative, the work done by the electric forces is positive $W = \frac{Q^2 s}{2\epsilon_0 L(L+x)^2}\Delta x$. *(iv)* For a small displacement in the positive x-direction, the work done is equal to $W = F\Delta x$. Therefore the horizontal force exerted by the charges on the outer plates acting on the charges on the middle plate is given by $F = \frac{Q^2 s}{2\epsilon_0 L(L+x)^2}$.

10. A flat conducting sheet A is suspended by an insulating thread between the surfaces formed by the bent conducting sheet B as shown in Fig. 12. The sheets are oppositely charged, the difference in potential, in volts, is ΔV. This causes a force F, in addition to the weight of A, pulling A downward. *(i)* What is the capacitance of this arrangement of conductors as a function of y, the distance that plate A is inserted between the sides of plate B ? *(ii)* How much energy is needed to increase the inserted distance by Δy? *(iii)* Find an expression for the difference in potential ΔV

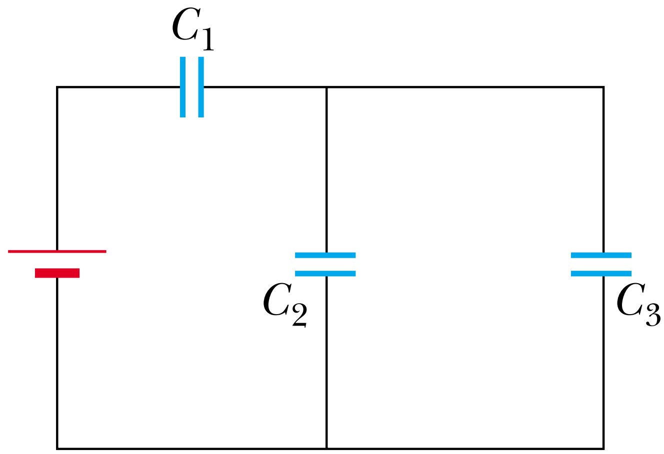

Figure 1: Problem 1.

in terms of F and relevant dimensions shown in the figure.

Solution: a) We begin by assuming the plates are very large and use Gauss law to calculate the electric field between the plates $\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\rm enc}}{\epsilon_0}$ Our choice of Gaussian surface is shown in Fig. 12. Then $\oiint \vec{E} \cdot d\vec{A} = EA_{\rm cap}$ and $\frac{q_{\rm enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$. Thus, Gauss' law implies that $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{\imath}$, between the plates. Note that the potential difference between the positive and negative plates is $V(+) - V(-) = -\int_{x=s}^{x=0} E_x dx = \frac{\sigma s}{\epsilon_0}$. So the surface charge density is equal to $\sigma = \frac{\epsilon_0[V(+)-V(-)]}{s}$. The area between the plates is yb. Note that there is a charge on inner surface of the surrounding sheet equal to σyb, so the total charge on the outer sheets is $Q = 2\sigma yb$. Therefore the capacitance is $C(y) = \frac{Q}{V(+)-V(-)} = \frac{2\sigma yb\epsilon_0}{\sigma s} = \frac{2\epsilon_0 yb}{s}$. *(ii)* If the inserted distance is increased by Δy then $\Delta C = \frac{2\epsilon_0 \Delta yb}{s}$. Because the charge on the plates is fixed, the energy stored in the capacitor is given by $U(y) = \frac{Q^2}{2C(y)}$. Hence, when the inserted distance is increased by Δy, the energy stored between the plates decreases by $\Delta U = \frac{dU}{dC}\Delta C = -\frac{Q^2}{2C^2}\Delta C = -\frac{[V(+)-V(-)]^2}{2}\frac{2\epsilon_0 \Delta yb}{s}$. *(iii)* This decrease in energy is used to pull the hanging plate in between the two positive charged plates. The work done in pulling the hanging plate a distance Δy is given by $\Delta W = F_y \Delta y$. By conservation of energy $0 = \Delta U + \Delta W = -\frac{[V(+)-V(-)]^2}{2}\frac{2\epsilon_0 \Delta yb}{s} + F_y \Delta y$. We can solve this equation for the y-component of the force $F_y = \frac{[V(+)-V(-)]^2}{2}\frac{2\epsilon_0 b}{s}$ or else find $V(+) - V(-) = \sqrt{\frac{sF_y}{\epsilon_0 b}}$.

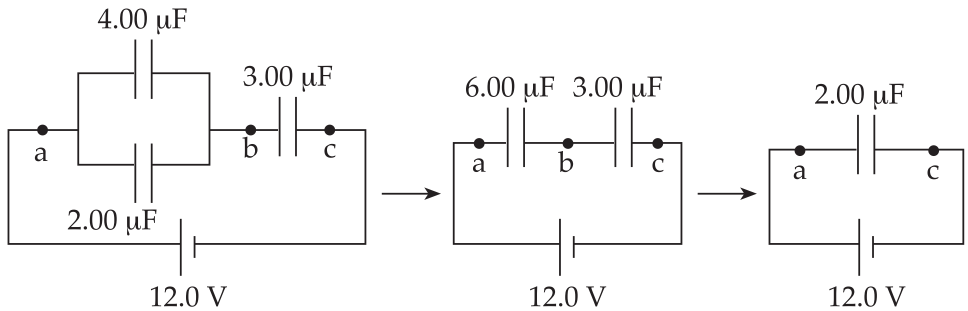

Figure 2: Solution of problem 1.

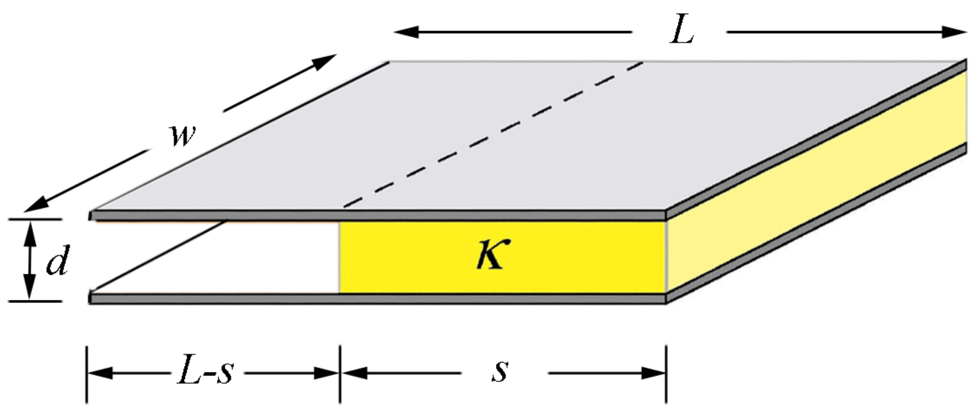

Figure 3: Problem 2.

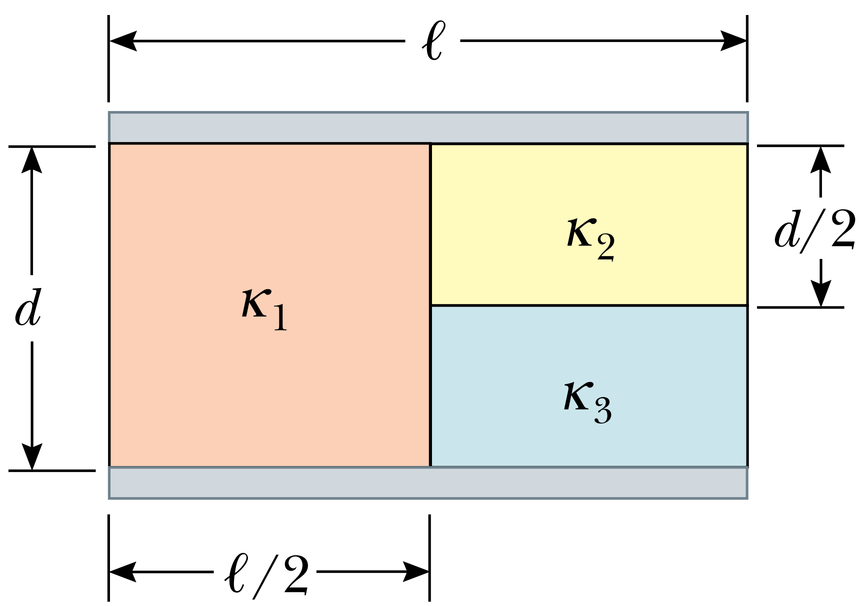

Figure 4: Problem 3.

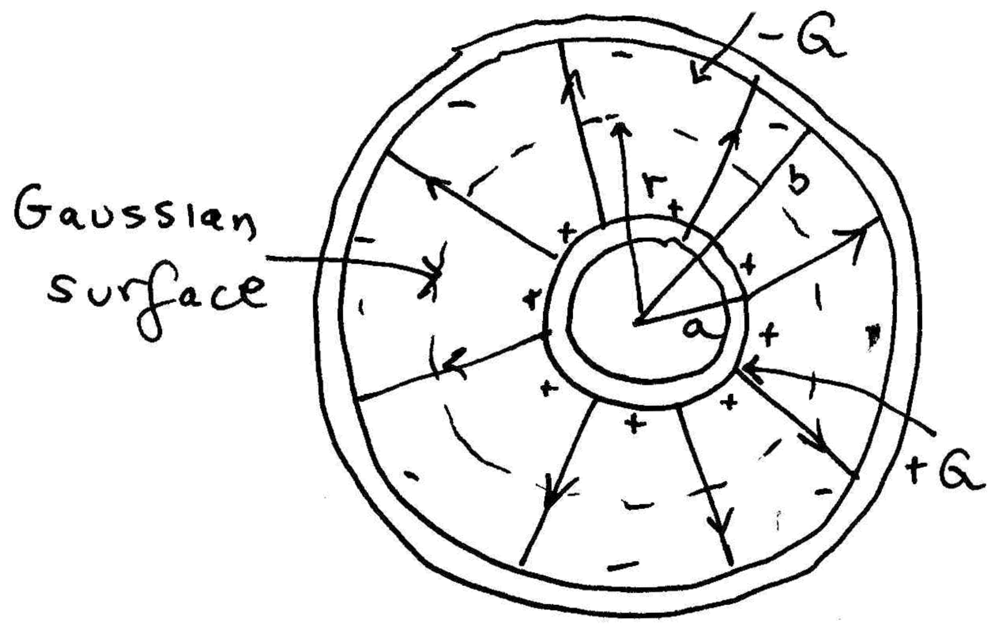

Figure 5: Problem 5.

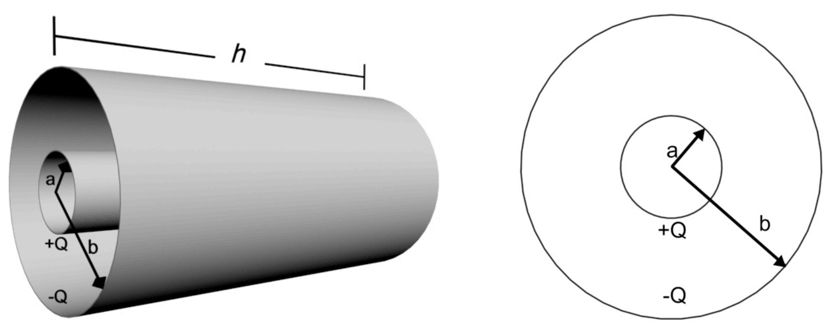

Figure 6: Problem 7.

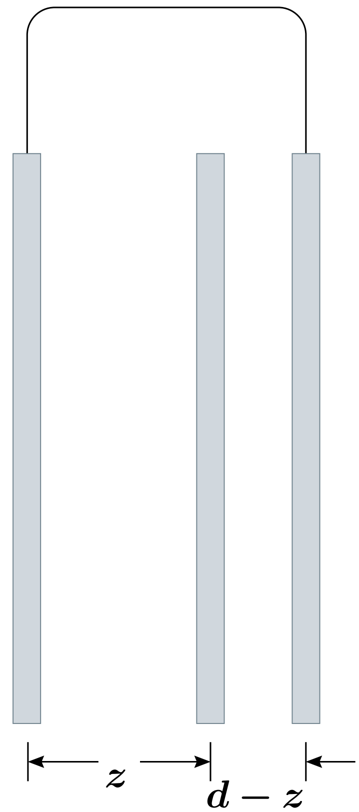

Figure 7: Problem 8.

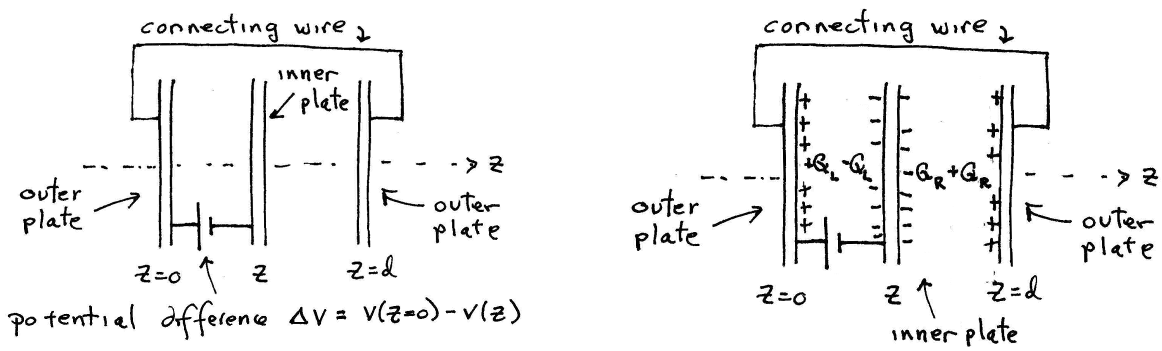

Figure 8: Solution of problem 8.

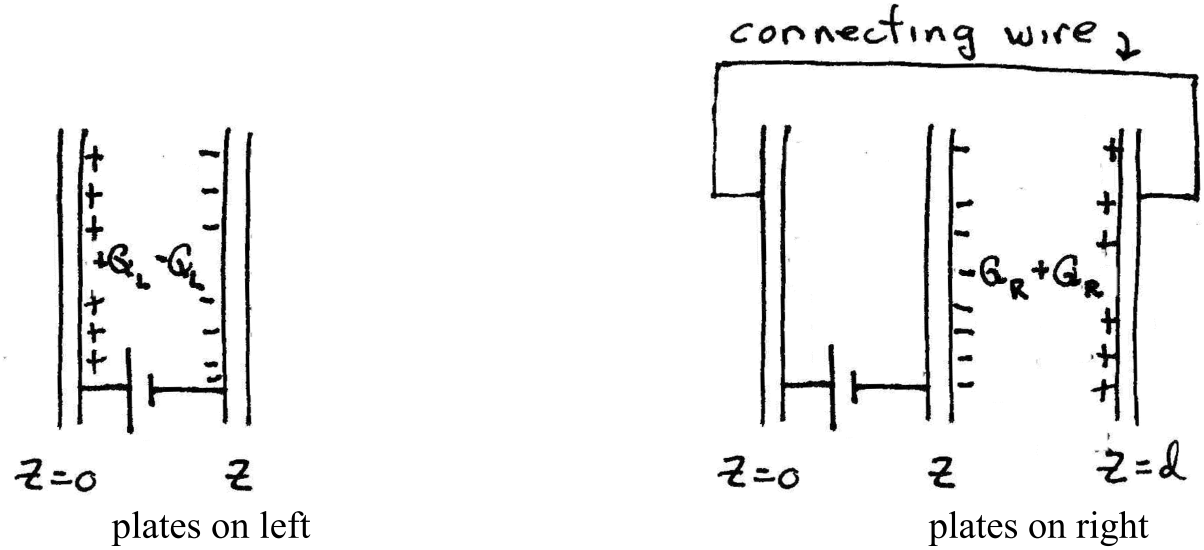

Figure 9: More on the solution of problem 8.

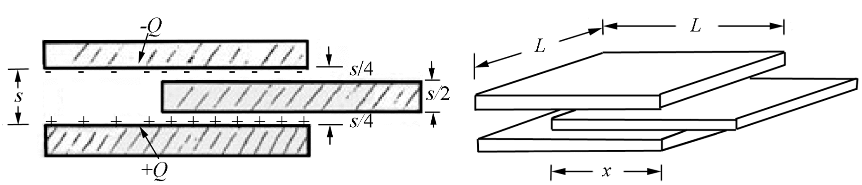

Figure 10: Problem 9.

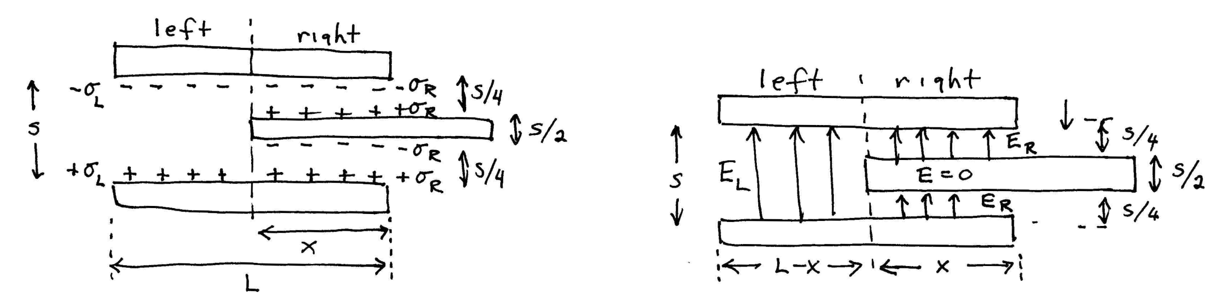

Figure 11: Solution of problem 9.

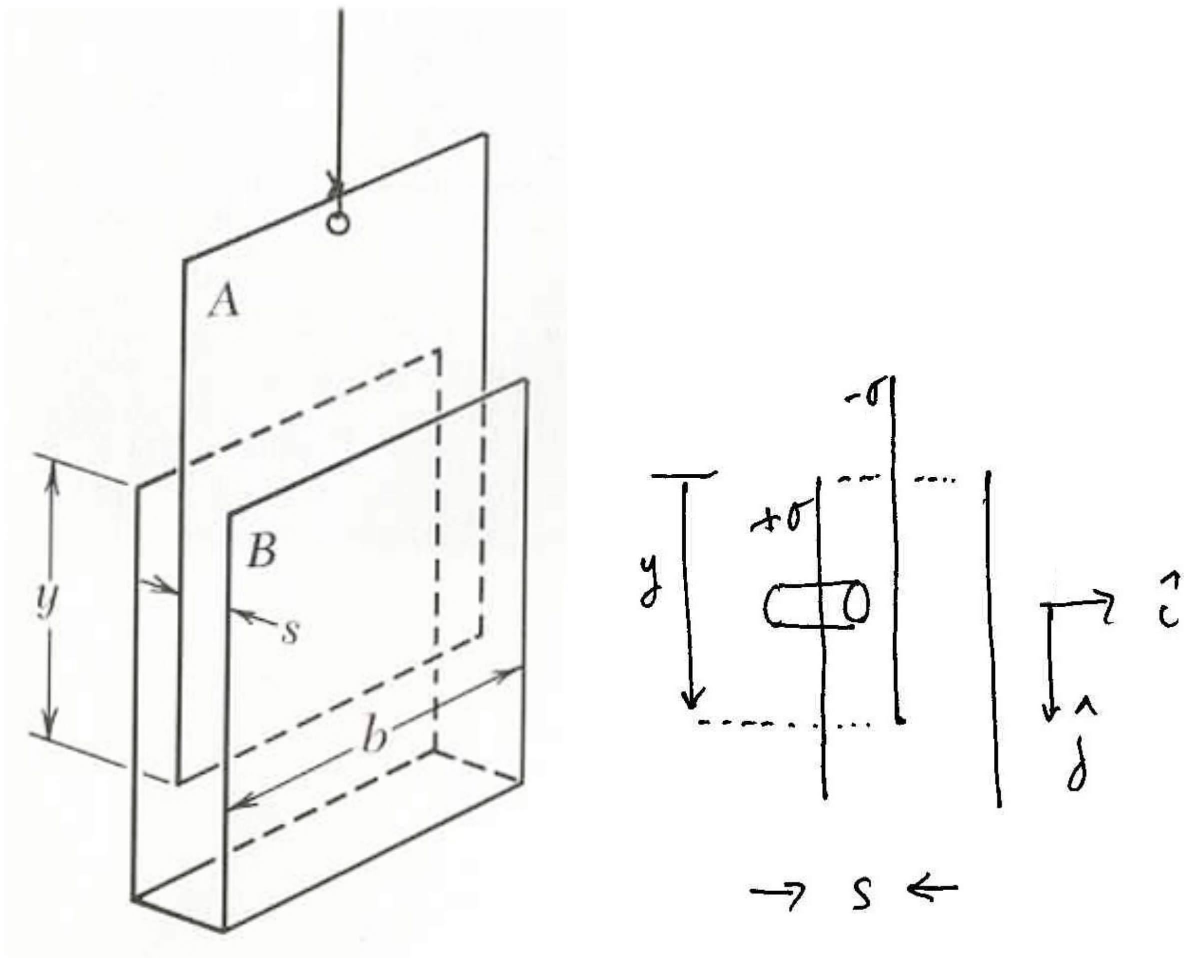

Figure 12: Problem 10.