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Problems set # 3 **Physics 169** February 24, 2015

1. A point charge q is located at the center of a uniform ring having linear charge density λ and radius a, as shown in Fig. 1. Determine the total electric flux through a sphere centered at the point charge and having radius R, where $R < a$.

Solution Only the charge inside radius R contributes to the total flux, hence $\Phi_E = q/\epsilon_0$.

2. A point charge Q is located just above the center of the flat face of a hemisphere of radius R as shown in Fig. 2. What is the electric flux *(i)* through the curved surface and *(ii)* through the flat face?

Solution With δ very small, all points on the hemisphere are nearly at a distance R from the charge, so the field everywhere on the curved surface is $\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$ radially outward (normal to the surface). Therefore, the flux is this field strength times the area of half a sphere $\Phi_{\rm curved} = \int \vec{E}\cdot d\vec{A} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}2\pi R^2 = \frac{Q}{2\epsilon_0}$. *(ii)* The closed surface encloses zero charge so Gauss' law gives $\Phi_{\rm curved} + \Phi_{\rm flat} = 0$ or $\Phi_{\rm flat} = -\Phi_{\rm curved} = -\frac{Q}{2\epsilon_0}$.

3. The line ag in Fig. 3 is a diagonal of a cube. A point charge q is located on the extension of line ag, very close to vertex a of the cube. Determine the electric flux through each of the sides of the cube which meet at the point a.

Solution No charge is inside the cube. The net flux through the cube is zero. Positive flux comes out through the three faces meeting at g. These three faces together fill a solid angle equal to one-eighth of a sphere as seen from q. The total flux passing through these faces is then $\frac{1}{8}\frac{q}{\epsilon_0}$. Each face containing a intercepts equal flux going into the cube: $0 = \Phi_{E,\rm net} = 3\Phi_{E,abcd} + \frac{q}{8\epsilon_0}$. Therefore, $\Phi_{E,abcd} = -\frac{q}{24\epsilon_0}$.

4. A sphere of radius R surrounds a point charge Q , located at its center. *(i)* Show that the electric flux through a circular cap of half-angle (see Fig. 4) is $\Phi_E = \frac{Q}{2\epsilon_0}(1-\cos\theta)$. What is the flux for *(ii)* $\theta = 90°$ and *(iii)* $\theta = 180°$.

Solution The charge creates a uniform $\vec{E}$, pointing radially outward, so $\Phi_E = EA$. The arc length is $ds = Rd\theta$, and the circumference is $2\pi r = 2\pi R\sin\theta$. Hence, $A = \int 2\pi r ds = \int_0^\theta (2\pi R\sin\theta)Rd\theta = 2\pi R^2\int_0^\theta \sin\theta\ d\theta = -2\pi R^2\cos\theta|_0^\theta = 2\pi R^2(1-\cos\theta)$ and $\Phi_E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\cdot 2\pi R^2(1-\cos\theta) = \frac{Q}{2\epsilon_0}(1-\cos\theta)$, i.e., independent of R! *(ii)* For $\theta = 90°$ (hemisphere), $\Phi_E = \frac{Q}{2\epsilon_0}(1-\cos 90°) = \frac{Q}{2\epsilon_0}$ *(iii)* For $\theta = 180°$ (entire sphere), $\Phi_E = \frac{Q}{2\epsilon_0}(1-\cos 180°) = \frac{Q}{\epsilon_0}$; this is a formal derivation of Gauss' law.

5. An insulating solid sphere of radius a has a uniform volume charge density and carries a

total positive charge Q. A spherical gaussian surface of radius r, which shares a common center with the insulating sphere, is inflated starting from $r = 0$. *(i)* Find an expression for the electric flux passing through the surface of the gaussian sphere as a function of r for $r < a$. *(ii)* Find an expression for the electric flux for $r > a$. *(iii)* Plot the flux versus r.

Solution The charge density is determined by $Q = \frac{4}{3}\pi a^{\rho} \Rightarrow \rho = \frac{3Q}{4\pi a^3}$. *(i)* The flux is that created by the enclosed charge within radius r: $\Phi_E = \frac{q_{\rm in}}{\epsilon_0} = \frac{4\pi r^3 \rho}{3\epsilon_0} = \frac{Qr^3}{\epsilon_0 a^3}$. *(ii)* $\Phi_E = \frac{Q}{\epsilon_0}$. Note that the answers to parts *(i)* and *(ii)* agree at $r = a$. *(iii)* This is shown in Fig. 5.

6. A solid insulating sphere of radius a carries a net positive charge $3Q$, uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius b and outer radius c, and having a net charge $-Q$, as shown in Fig. 6. *(i)* Construct a spherical gaussian surface of radius $r > c$ and find the net charge enclosed by this surface. *(ii)* What is the direction of the electric field at $r > c$? *(iii)* Find the electric field at $r \geq c$. *(iv)* Find the electric field in the region with radius r where $b < r < c$ *(v)* Construct a spherical gaussian surface of radius r, where $b < r < c$, and find the net charge enclosed by this surface. *(vi)* Construct a spherical gaussian surface of radius r, where $a < r < b$, and find the net charge enclosed by this surface. *(vii)* Find the electric field in the region $a < r < b$. *(viii)* Construct a spherical gaussian surface of radius $r < a$, and find an expression for the net charge enclosed by this surface, as a function of r. Note that the charge inside this surface is less than $3Q$. *(ix)* Find the electric field in the region $r < a$. *(x)* Determine the charge on the inner surface of the conducting shell. *(xi)* Determine the charge on the outer surface of the conducting shell. *(xii)* Make a plot of the magnitude of the electric field versus r.

Solution *(i)* $q_{\rm in} = 3Q - Q = 2Q$. *(ii)* The charge distribution is spherically symmetric and $q_{\rm in} > 0$. Thus, the field is directed radially outward. *(iii)* For $r \geq c$, $E = \frac{1}{4\pi\epsilon_0}\frac{q_{\rm in}}{r^2} = \frac{Q}{2\pi\epsilon_0 r^2}$. *(iv)* Since all points within this region are located inside conducting material, $E = 0$, for $b < r < c$. *(v)* $\Phi_E = \int \vec{E} \cdot d\vec{A} = 0 \Rightarrow q_{\rm in} = \epsilon_0 \Phi_E = 0$. *(vi)* $q_{\rm in} = 3Q$. *(vii)* For $a \leq r < b$, $E = \frac{1}{4\pi\epsilon_0}\frac{q_{\rm in}}{r^2} = \frac{3Q}{4\pi\epsilon_0 r^2}$ (radially outward). *(viii)* $q_{\rm in} = \rho V = \frac{3Q}{\frac{4}{3}\pi a^3}\frac{4}{3}\pi r^3 = 3Q\frac{r^3}{a^3}$. *(ix)* For $0 \leq r \leq a$, $E = \frac{q_{\rm in}}{4\pi\epsilon_0 r^2} = \frac{3Qr}{4\pi\epsilon_0 a^3}$ (radially outward). *(x)* From part *(iv)*, for $b < r < c$, $E = 0$.Thus, for a spherical gaussian surface with $b < r < c$, $q_{\rm in} = 3Q + q_{\rm inner} = 0$ where $q_{\rm inner}$ is the charge on the inner surface of the conducting shell. This yields $q_{\rm inner} = -3Q$. *(xi)* Since the total charge on the conducting shell is $q_{\rm net} = q_{\rm outer} + q_{\rm inner} = -Q$, we have $q_{\rm outer} = -Q - q_{\rm inner} = -Q - (-3Q) = 2Q$. *(xii)* This is shown in Fig. 6.

7.Consider a long cylindrical charge distribution of radius R with a uniform charge density ρ. Find the electric field at distance r from the axis where $r < R$.

Solution If is positive, the field must be radially outward. Choose as the gaussian surface a cylinder of length L and radius r, contained inside the charged rod. Its volume is $\pi r^2 L$ and it encloses charge $\rho\pi r^2 L$, see Fig. 7. Because the charge distribution is long, no electric flux passes through the circular end caps; $\vec{E} \cdot d\vec{A} = E dA \cos(\pi/2) = 0$. The curved surface has $\vec{E} \cdot d\vec{A} = E dA \cos 0°$, and E must be the same strength everywhere over the curved surface. Gauss' law, $\oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$, becomes $E\int_{\substack{\text{curved}\\\text{surface}}} dA = \frac{\rho\pi r^2 L}{\epsilon_0}$. Now the lateral surface area of the cylinder is $2\pi rL$, yielding $E2\pi rL = \rho\pi r^2 L/\epsilon_0$. Thus, $\vec{E} = \frac{\rho r}{2\epsilon_0}$ radially away from the cylinder axis.

8. A solid, insulating sphere of radius a has a uniform charge density ρ and a total charge Q. Concentric with this sphere is an uncharged, conducting hollow sphere whose inner and outer radii are b and c, as shown in Fig. 8. *(i)* Find the magnitude of the electric field in the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$. *(ii)* Determine the induced charge per unit area on the inner and outer surfaces of the hollow sphere.

Solution *(i)* From Gauss' law $\oint \vec{E}\cdot d\vec{A} = E(4\pi r^2) = \frac{q_{\rm in}}{\epsilon_0}$. For $r < a$, $q_{\rm in} = \rho\frac{4}{3}\pi r^3$, so $E = \frac{\rho r}{3\epsilon_0}$. For $a < r < b$ and $c < r$, $q_{\rm in} = Q$, so $E = \frac{Q}{4\pi\epsilon_0 r^2}$. For $b \leq r \leq c$, $E = 0$, since $E = 0$ inside a conductor. *(ii)* Let q_1 be the induced charge on the inner surface of the hollow sphere. Since $E = 0$ inside the conductor, the total charge enclosed by a spherical surface of radius $b \leq r \leq c$ must be zero. Therefore, $q_1 + Q = 0$ and $\sigma_1 = \frac{q_1}{4\pi b^2} = -\frac{Q}{4\pi b^2}$. Let q_2 be the induced charge on the outside surface of the hollow sphere. Since the hollow sphere is uncharged, we require $q_1 + q_2 = 0$ and $\sigma_2 = \frac{q_1}{4\pi c^2} = \frac{Q}{4\pi c^2}$.

9. An early (incorrect) model of the hydrogen atom, suggested by J. J. Thomson, proposed that a positive cloud of charge e was uniformly distributed throughout the volume of a sphere of radius R, with the electron an equal-magnitude negative point charge e at the center. *(i)* Using Gauss' law, show that the electron would be in equilibrium at the center and, if displaced from the center a distance $r < R$, would experience a restoring force of the form $F = -kr$, where k is a constant. *(ii)* Show that $k = \frac{e^2}{4\pi\epsilon_0 R^3}$. *(iii)* Find an expression for the frequency f of simple harmonic oscillations that an electron of mass m_e would undergo if displaced a small distance ($< R$) from the center and released. *(iv)* Calculate a numerical value for R that would result in a frequency of 2.47×10^{15} Hz, the frequency of the light radiated in the most intense line in the hydrogen spectrum.

Solution First, consider the field at distance $r < R$ from the center of a uniform sphere of positive charge ($Q = +e$) with radius R. From Gauss' law, $4\pi r^2 E = \frac{q_{\rm in}}{\epsilon_0} = \frac{\rho V}{\epsilon_0} = \frac{+e}{\frac{4}{3}\pi R^3}\frac{\frac{4}{3}\pi r^2}{\epsilon_0}$, so $E = \frac{e}{4\pi\epsilon_0 R^3} r$ directed outward. *(i)* The force exerted on a point charge $q = -e$ located at distance r from the center is then $F = qE = -\frac{e^2}{4\pi\epsilon_0 R^3} r = -kr$. *(ii)* $k = \frac{e^2}{4\pi\epsilon_0 R^3}$ *(iii)* $F_r = m_e a_r = -\frac{e^2}{4\pi\epsilon_0 R^3} r$, so $a_r = -\frac{e^2}{4\pi\epsilon_0 m_e R^3} r = -\omega^2 r$. Thus, the motion is simple harmonic with frequency $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{e^2}{4\pi\epsilon_0 m_e R^3}}$. *(iv)* $f = 2.47\times 10^{15}$ Hz $= \frac{1}{2\pi}\sqrt{\frac{8.99\times10^9 \text{N}\cdot\text{m}^2/\text{C}^2 (1.60\times10^{-19}\ \text{C})^2}{9.11\times10^{-31}\ \text{kg}\ R^3}}$, which yields $R^3 = 1.05\times 10^{-30}$ m^3, or $R = 1.02\times 10^{-10}$ m $= 102$ pm.

10. An infinitely long cylindrical insulating shell of inner radius a and outer radius b has a uniform volume charge density ρ. A line of uniform linear charge density λ is placed along the axis of the shell. Determine the electric field everywhere.

Solution The field direction is radially outward perpendicular to the axis. The field strength depends on r but not on the other cylindrical coordinates θ or z. Choose a Gaussian cylinder of radius r and length L. If $r < a$, $\Phi_E = \frac{q_{\rm in}}{\epsilon_0}$ and $E2\pi rL = \lambda L/\epsilon_0$, so $\vec{E} = \frac{\lambda}{2\pi r\epsilon_0}\hat{r}$. If $a < r < b$, $E2\pi rL = [\lambda L + \rho\pi(r^2 - a^2)L]/\epsilon_0$ and $\vec{E} = \frac{\lambda+\rho\pi(r^2-a^2)}{2\pi r\epsilon_0}\hat{r}$. If $r > b$, $E2\pi rL = [\lambda L + \rho\pi(b^2 - a^2)L]/\epsilon_0$ and $\vec{E} = \frac{\lambda+\rho\pi(b^2-a^2)}{2\pi r\epsilon_0}\hat{r}$.

11. A particle of mass m and charge q moves at high speed along the x axis. It is initially near $x = -\infty$, and it ends up near $x = +\infty$. A second charge Q is fixed at the point $x = 0$, $y = -d$. As the moving charge passes the stationary charge, its x component of velocity does not change appreciably, but it acquires a small velocity in the y direction. Determine the angle through which the moving charge is deflected. [*Hint:* The integral you encounter in determining v_y can be evaluated by applying Gauss' law to a long cylinder of radius d, centered on the stationary charge.]

Solution The vertical velocity component of the moving charge increases according to $m\frac{dv_y}{dt} = F_y \to m\frac{dv_y}{dx}\frac{dx}{dt} = qE_y$, see Fig. 9. Now $\frac{dx}{dt} = v_x$, has a nearly constant value v. Hence $dv_y = \frac{q}{mv}E_y dx$, yielding $v_y = \int_0^{v_y} dv'_y = \frac{q}{mv}\int_{-\infty}^{+\infty} E_y dx$. The radially outward compnent of the electric field varies along the x axis; it is described by $\int_{-\infty}^{+\infty} E_y dA = \int_{-\infty}^{+\infty} E_y 2\pi d\, dx = Q/\epsilon_0$. Putting all this together $\int_{-\infty}^{+\infty} E_y dx = \frac{Q}{2\pi d\epsilon_0}$ and $v_y = \frac{qQ}{mv2\pi d\epsilon_0}$. The angle of deflection is described by $\tan\theta = v_y/v$, so $\theta = \tan^{-1}\frac{qQ}{2\pi\epsilon_0 dmv^2}$.

12. Two infinite, nonconducting sheets of charge are parallel to each other, as shown in Fig. 10. The sheet on the left has a uniform surface charge density σ, and the one on the right has a uniform charge density $-\sigma$. Calculate the electric field at points *(i)* to the left of, *(ii)* in between, and *(iii)* to the right of the two sheets. *(iv)* Repeat the calculations when both sheets have positive uniform surface charge densities of value σ.

Solution Consider the field due to a single sheet and let E_+ and E_- represent the fields due to the positive and negative sheets, see Fig. 10. The field at any distance from each sheet has a magnitude given by $|E_+| = |E_-| = \frac{\sigma}{2\epsilon_0}$. *(i)* To the left of the positive sheet, E_+ is directed toward the left and E_- toward the right and the net field over this region is $\vec{E} = 0$. *(ii)* In the region between the sheets, E_+ and E_- are both directed toward the right and the net field is $\vec{E} = \sigma/\epsilon_0$ to the right. *(iii)* To the right of the negative sheet, E_- and E_+ are again oppositely directed and $\vec{E} = 0$. *(iv)* If both charges are positive (see Fig. 10), in the region to the left of the pair of sheets, both fields are directed toward the left and the net field is $\vec{E} = \sigma/\epsilon_0$ to the left; in the region between the sheets, the fields due to the individual sheets are oppositely directed and the net field is $\vec{E} = 0$; in the region to the right of the pair of sheets, both are fields are directed toward the right and the net field is $\vec{E} = \sigma/\epsilon_0$ to the right.

13. A sphere of radius $2a$ is made of a nonconducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius a is now removed from the sphere, as shown in Fig. 11. Show that the electric field within the cavity is uniform and is given by $E_x = 0$ and $E_y = \frac{\rho a}{3\epsilon_0}$. [*Hint*: The field within the cavity is the superposition of the field due to the original uncut sphere, plus the field due to a sphere the size of the cavity with a uniform negative charge density $-\rho$].

Solution The resultant field within the cavity is the superposition of two fields, one $\vec{E}_+$ due to a uniform sphere of positive charge of radius $2a$, and the other $\vec{E}_-$ due to a sphere of negative charge of radius a centered within the cavity. From Gauss' law we have $\frac{4}{3}\frac{\pi r^3 \rho}{\epsilon_0} = 4\pi r^2 E_+$, so $\vec{E}_+ = \frac{\rho r}{3\epsilon_0}\hat{r} = \frac{\rho}{3\epsilon_0}\vec{r}$. Using again Gauss' law, $-\frac{4}{3}\frac{\pi r_1^3 \rho}{\epsilon_0} = 4\pi r_1^2 E_-$, so $\vec{E}_- = \frac{\rho r_1}{3\epsilon_0}(-\hat{r}_1) = -\frac{\rho}{3\epsilon_0}\vec{r}_1$. It is easily seen in Fig. 11 that $\vec{r} = a + \vec{r}_1$, so $E_= - \frac{\rho(\vec{r}-\vec{a})}{3\epsilon_0}$, yielding $\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho\vec{r}}{3\epsilon_0} - \frac{\rho\vec{r}}{3\epsilon_0} + \frac{\rho\vec{a}}{3\epsilon_0} = \frac{\rho\vec{a}}{3\epsilon_0} = 0\hat{\imath} + \frac{\rho a}{3\epsilon_0}\hat{\jmath}$. Therefore $E_x = 0$ and $E_y = \frac{\rho a}{3\epsilon_0}$ at all points within the cavity.

14. A solid insulating sphere of radius R has a nonuniform charge density that varies with r according to the expression $\rho = Ar^2$, where A is a constant and $r < R$ is measured from the center of the sphere. *(i)* Show that the magnitude of the electric field outside ($r > R$) the sphere is $E = \frac{AR^5}{5\epsilon_0 r^2}$. *(ii)* Show that the magnitude of the electric field inside ($r < R$) the sphere is $E = \frac{Ar^3}{5\epsilon_0}$. [*Hint*: The total charge Q on the sphere is equal to the integral of ρdV, where r extends from 0 to R; also, the charge q within a radius $r < R$ is less than Q. To evaluate the integrals, note that the volume element dV for a spherical shell of radius r and thickness dr is equal to $4r^2 dr$.]

Solution From Gauss' law $\oint \vec{E} \cdot d\vec{A} = E 4\pi r^2 = q_{\rm in}/\epsilon_0$. *(i)* For $r > R$, $q_{\rm in} = \int_0^R Ar^2 4\pi r^2\, dr = 4\pi AR^5/5$, and $E = \frac{AR^5}{5\epsilon_0 r^2}$. *(ii)* For $r < R$, $q_{\rm in} = \int_0^r Ar^2 4\pi r^2 dr = 4\pi Ar^5/5$, and $E = \frac{Ar^3}{5\epsilon_0}$.

15. A slab of insulating material (infinite in two of its three dimensions) has a uniform positive charge density ρ. An edge view of the slab is shown in Fig.12. *(i)* Show that the magnitude of the electric field a distance x from its center and inside the slab is $E = \rho x/\epsilon_0$. *(ii)* Suppose an electron of charge $-e$ and mass m_e can move freely within the slab. It is released from rest at a distance x from the center. Show that the electron exhibits simple harmonic motion with a frequency $s = \frac{1}{2\pi}\sqrt{\frac{\rho e}{m_e \epsilon_0}}$. *(iii)* A slab of insulating material has a nonuniform positive charge density $\rho = Cx^2$, where x is measured from the center of the slab as shown in Fig. 12, and C is a constant. The slab is infinite in the y and z directions. Derive expressions for the electric field in the exterior regions and the interior region of the slab ($-d/2 < x < d/2$).

Solution *(i)* Consider a cylindrical shaped gaussian surface perpendicular to the yz plane with one end in the yz plane and the other end containing the point x: Use Gauss law: $\oint \vec{E} \cdot d\vec{A} = q_{\rm in}/\epsilon_0$ By symmetry, the electric field is zero in the yz plane and is perpendicular to $d\vec{A}$ over the wall of the gaussian cylinder. Therefore, the only contribution to the integral is over the end cap containing the point x. Hence $EA = \rho Ax/\epsilon_0$, so that at distance x from the mid-line of the slab, $E = \rho x/\epsilon_0$. *(ii)* Use Newton's law to obtain $a = \frac{F}{m_e} = -\frac{\rho e}{m_e \epsilon_0}x$. The acceleration of the electron is of the form $a = -\omega^2 x$ with $\omega = \sqrt{\frac{\rho e}{m_e \epsilon_0}}$. Thus, the motion is simple harmonic with frequency $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\rho e}{m_e \epsilon_0}}$.

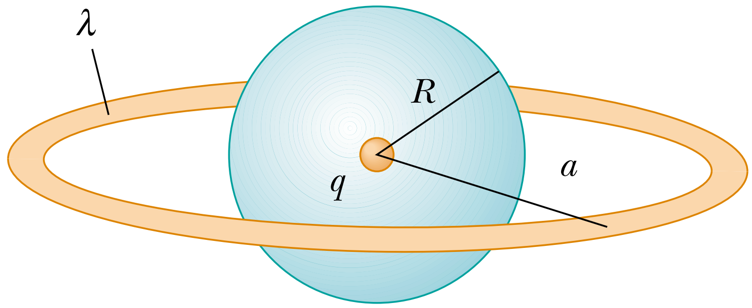


Figure 1: Problem 1.

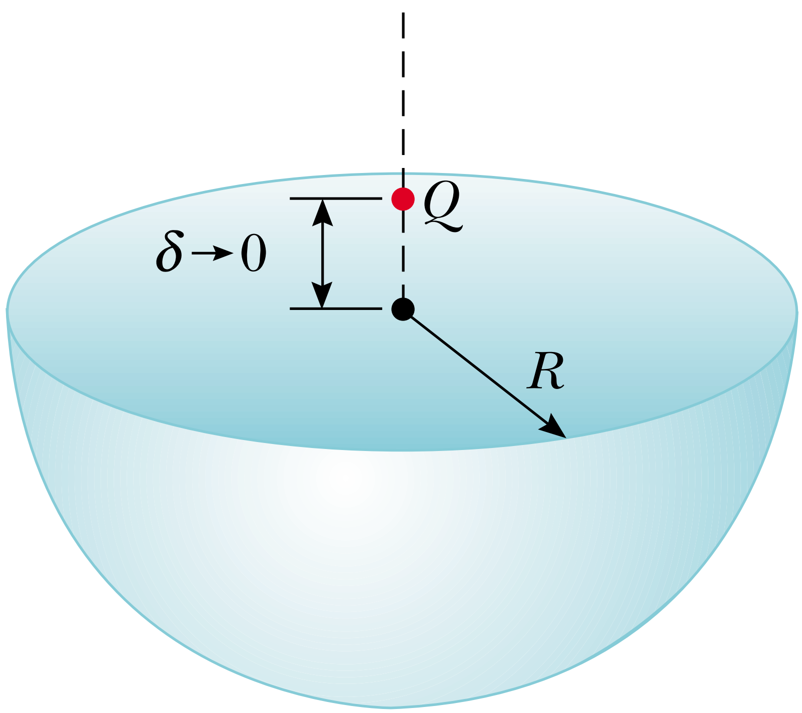


Figure 2: Problem 2.

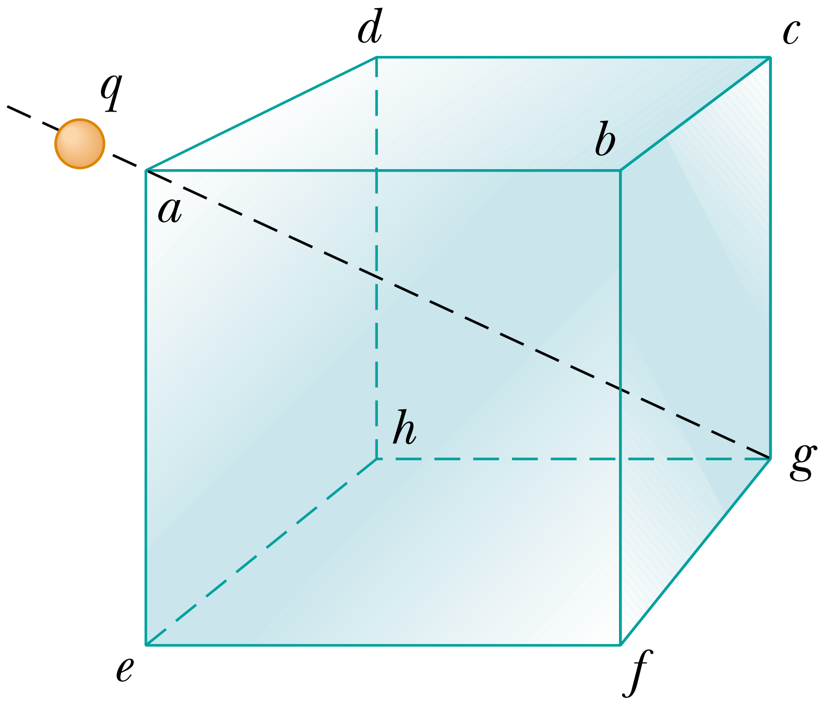
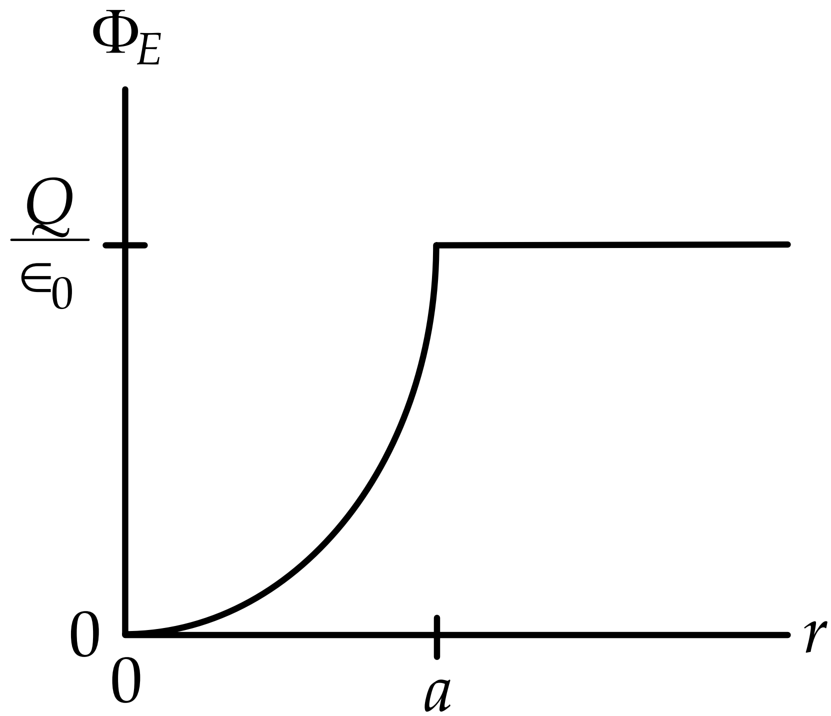


Figure 3: Problem 3.

Figure 4: Problem 4.



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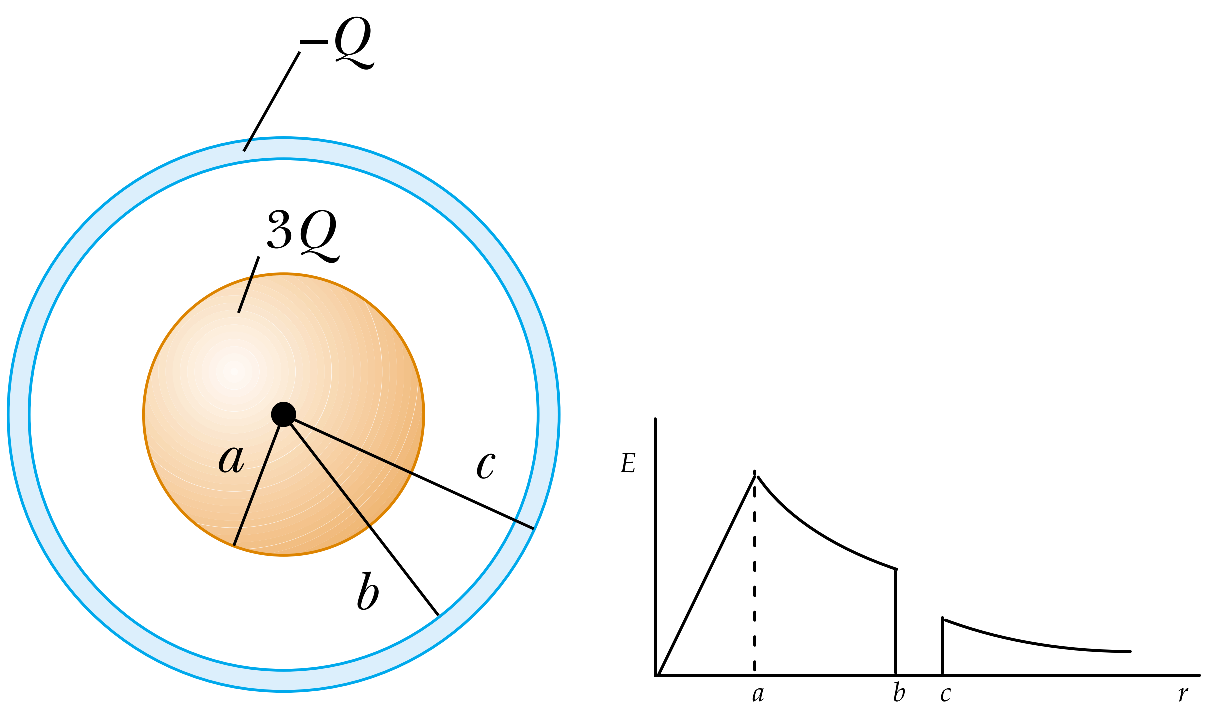

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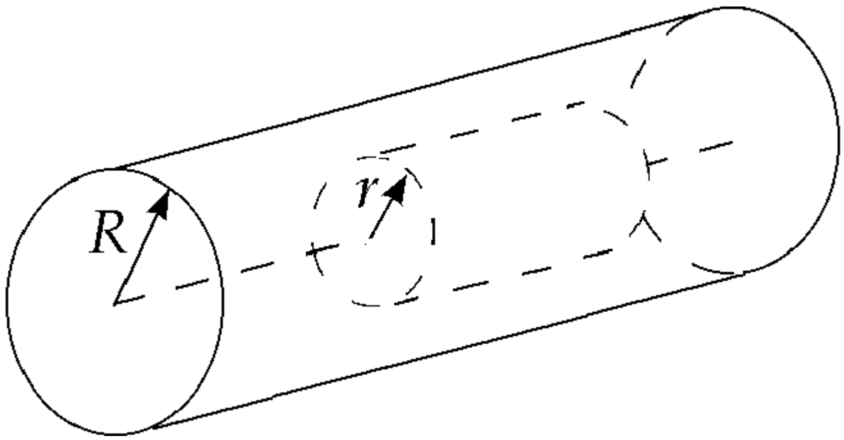

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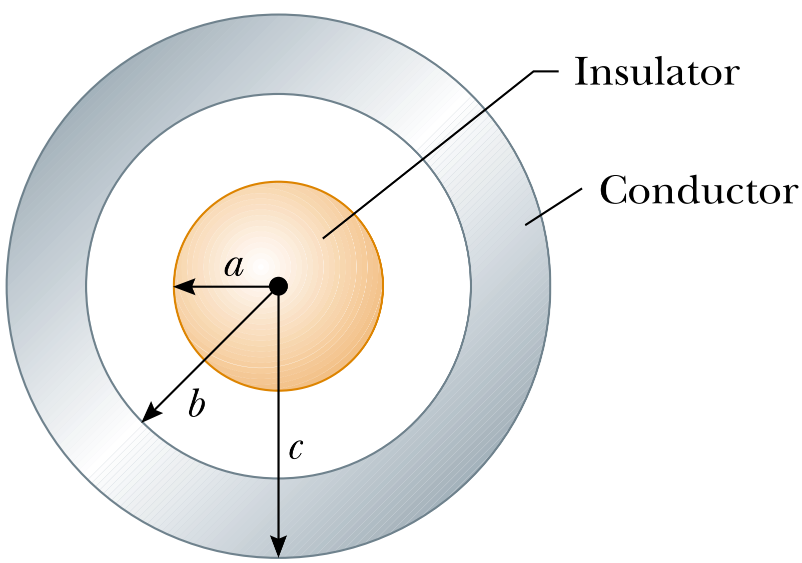

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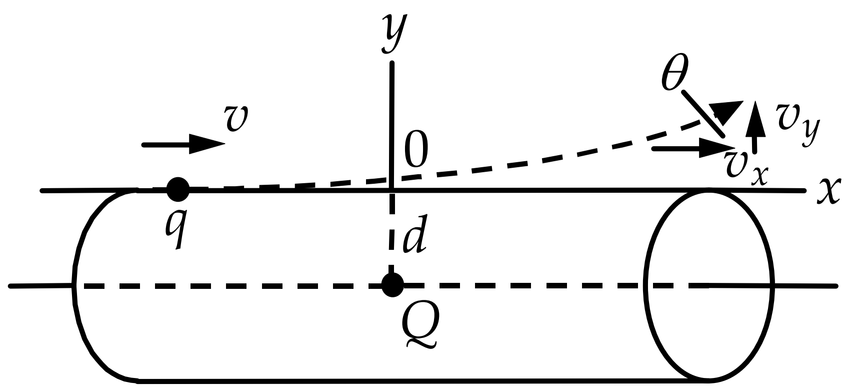

Figure 9: Problem 11.

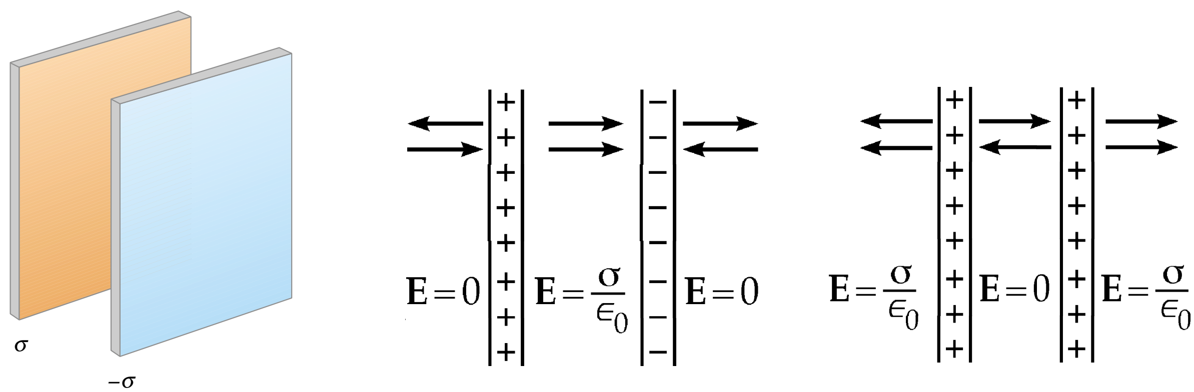

Figure 10: Problem 12.

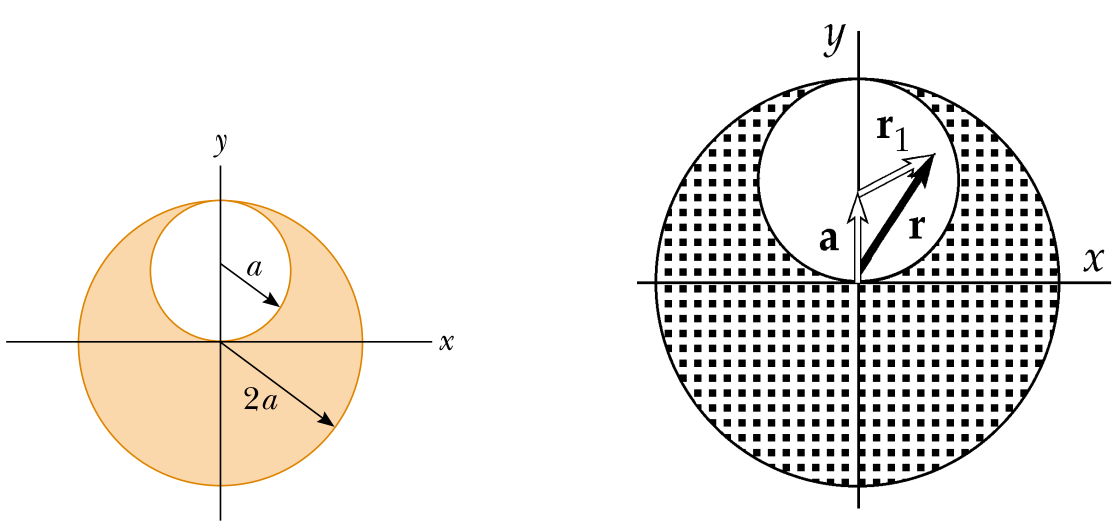

Figure 11: Problem 13.

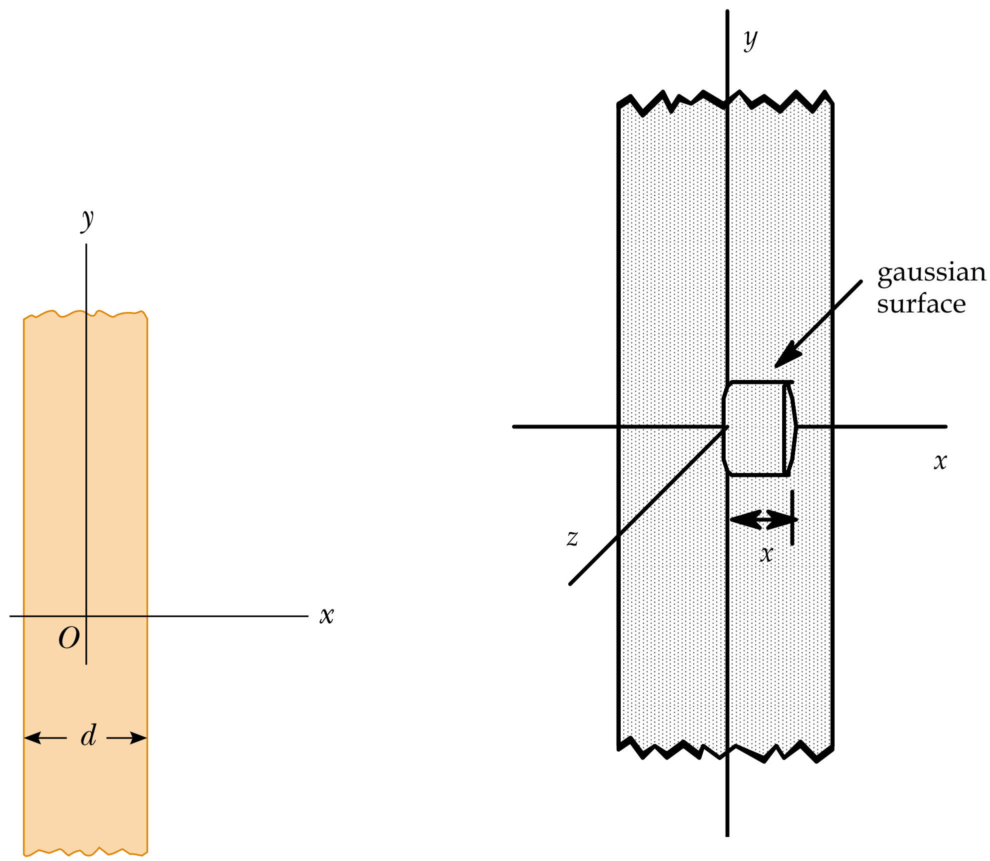

Figure 12: Problem 15.