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Problems set # 12 **Physics 169** May 12, 2015

1. Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ϵ/ϵ_0), as shown in Fig. 1. *(i)* Find the electric field everywhere between the spheres. *(ii)* Calculate the surface-charge distribution on the inner sphere. *(iii)* Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.

Solution This is a somewhat curious problem. It should be obvious that without any dielectric the electric field between the spheres would be radial $\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$. We cannot expect this to be unmodified by the dielectric. However, we note that the radial electric field is tangential to the interface between the dielectric and empty region. Thus the tangential matching condition $E_1^{\|} = E_2^{\|}$ is automatically satisfied. At the same time there is no perpendicular component to the interface, so there is nothing to worry about for the $D_1^{\perp} = D_2^{\perp}$ matching condition. This suggests that we guess a solution of the radial form $E = A\hat{r}/r^2$, where A is a constant to be determined. This guess is perhaps not completely obvious because one may have expected the field lines to bend into or out of the dielectric region. However, we could also recall that parallel fields do not get bent across the dielectric interface. We may use the integral form of Gauss law in a medium to determine the above constant A, i.e., $\oiint \vec{D}\cdot\hat{n}dA = Q \Rightarrow \frac{\epsilon_0 A}{r^2}2\pi r^2 + \frac{\epsilon A}{r^2}2\pi r^2 = Q$, or $A = \frac{Q}{2\pi(\epsilon+\epsilon_0)}$. Hence, $\vec{E} = \frac{Q}{2\pi(\epsilon+\epsilon_0)}\frac{\hat{r}}{r^2}$. Note that $(\epsilon+\epsilon_0)/2$ may be viewed as the average permittivity in the volume between the spheres. *(ii)* The surface-charge density is given by $\sigma = D^{\perp}|_{r=a}$, where either $D^{\perp} = \epsilon_0 E^{\perp}$ or $D^{\perp} = \epsilon E^{\perp}$ depending on region. This gives

$$\sigma = \begin{cases} \frac{\epsilon}{\epsilon+\epsilon_0}\frac{Q}{2\pi a^2}; & \text{dielectric side} \\ \frac{\epsilon}{\epsilon_0+\epsilon_0}\frac{Q}{2\pi a^2}; & \text{empty side} \end{cases} . \tag{1}$$

Note that the total charge obtained by integrating σ over the surface of the inner sphere gives Q as expected. *(iii)* The polarization charge density is given by $\rho_{\rm pol} = -\vec{\nabla}\cdot\vec{P}$, where $\vec{P} = \epsilon_0\chi_e\vec{E} = (\epsilon-\epsilon_0)\vec{E}$. Since the surface of the dielectric at $r = a$ is against the inner sphere, we can take the polarization to be zero inside the metal ("outside" the dielectric). Gauss' law in this case gives $\sigma_{\rm pol} = -P^{\perp}|_{r=a} = -(\epsilon-\epsilon_0)E^{\perp}|_{r=a} = -\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{Q}{2\pi a^2}$. Note that when this is combined with (1), the total (free and polarization) charge density is $\sigma_{\rm tot} = \sigma + \sigma_{\rm pol} = \frac{\epsilon_0}{\epsilon+\epsilon_0}\frac{Q}{2\pi a^2}$ on either half of the sphere. Since this is uniform, this is why the resulting electric field is radially symmetric.

2. A coaxial capacitor of length $l = 6$ cm uses an insulating dielectric material with $\epsilon/\epsilon_0 = 9$, see Fig. 2. The radii of the cylindrical conductors are 0.5 cm and 1 cm. If the voltage applied across the capacitor is $V(t) = 50\sin(120\pi t)$ what is the displacement current?

Solution To find the displacement current, we need to know $\vec{E}$ in the dielectric space between the cylindrical conductors. Using Gauss law we obtain $\vec{E} = -\frac{Q}{2\pi\epsilon r l}\hat{r}$ and so $V = \frac{Q}{2\pi\epsilon l}\ln(a/b)$. Therefore, $\vec{E} = -\frac{V}{r\ln(a/b)}\hat{r} = \frac{50\sin(120\pi t)}{r\ln 2}\hat{r} = -\frac{72.1}{r}\sin(120\pi t)\hat{r}$ V/m. This implies that $\vec{D} = \epsilon E =$

$\kappa\epsilon_0 E = -9\cdot 8.85\times 10^{-12}\cdot \frac{72.1}{r}\sin(120\pi t)\hat{r} = -\frac{5.75\times 10^{-9}}{r}\sin(120\pi t)\hat{r}$ C/m^2. The displacement current flows between the conductors through an imaginary cylindrical surface of length l and radius r. The current flowing from the outer conductor to the inner conductor along $-\hat{r}$ crosses surface $\vec{S}$ where $\vec{S} = -2\pi r l\hat{r}$. Therefore, $I_d = \frac{\partial \vec{D}}{\partial t}\cdot\vec{S} = -\frac{\partial}{\partial t}\left[\frac{5.75\times 10^{-9}}{r}\sin(120\pi t)\right]\hat{r}\cdot(-\hat{r}2\pi r l) = 5.75\times 10^{-9}\cdot 120\pi\cdot 2\pi l\cos(120\pi t) = 0.82\cos(120\pi t)$ μA. Alternatively, since the coaxial capacitor is lossless, its displacement current has to be equal to the conduction current flowing through the wires connected to the voltage sources. The capacitance of a coaxial capacitor is given by $C = \frac{2\pi\epsilon l}{\ln(a/b)}$. The current is $I = C\frac{dV}{dt} = \frac{2\pi\epsilon l}{\ln(b/a)}[120\pi\times 50\cos(120\pi t)] = 0.82\cos(120\pi t)$ μA, which is the same answer we obtained before.

3. The parallel-plate capacitor shown in Fig. 3 is filled with a lossy dielectric material of relative permittivity κ and conductivity σ. The separation between the plates is d and each plate is of area A. The capacitor is connected to a time-varying voltage source $V(t)$. *(i)* Obtain an expression for I_c, the conduction current flowing between the plates inside the capacitor, in terms of the given quantities. *(ii)* Obtain an expression for I_d, the displacement current flowing inside the capacitor. *(iii)* Based on your expressions for parts (i) and (ii), give an equivalent-circuit representation for the capacitor. *(iv)* Evaluate the values of the circuit elements for $A = 4$ cm^2, $d = 0.5$ cm, $\kappa = 4$, $\sigma = 2.5$ (S/m), and $V(t) = 10\cos(3\pi 10^3 t)$ V. [*Hint*: $1\ S = 1\Omega^{-1}$, (S stands for siemens)]

<u>Solution</u> The resistance is $R = \frac{d}{\sigma A}$, and so $I_c = \frac{V}{R} = \frac{V\sigma A}{d}$. *(ii)* The electric field is $E = V/d$ and so $I_d = \frac{\partial D}{\partial t}A = \epsilon A\frac{\partial E}{\partial t} = \frac{\epsilon A}{d}\frac{\partial V}{\partial t}$. *(iii)* The conduction current is directly proportional to V, as characteristic of a resistor, whereas the displacement current varies as $\partial V/\partial t$, which is characteristic of a capacitor. Hence, $R = \frac{d}{\sigma A}$ and $C = \frac{\epsilon A}{d}$. The circuit is shown in Fig. 3. *(iv)* $R = \frac{0.5\times 10^{-2}}{2.5\cdot 4\times 10^{-4}} = 5\ \Omega$ and $C = \frac{4\cdot 8.85\times 10^{-12}\cdot 4\times 10^{-4}}{0.5\times 10^{-2}} = 2.84\times 10^{-12}$ F.

4. Figure 4 shows a plane electromagnetic sinusoidal wave propagating in the x-direction. Suppose that the wavelength is 50 m, and the electric field vibrates in the xy plane with an amplitude of 22 V/m. Calculate *(i)* the frequency of the wave and *(ii)* the magnitude and direction of the magnetic field when the electric field has its maximum value in the negative y-direction. *(iii)* Write an expression for the magnetic field with the correct unit vector, with numerical values for $B_{\rm max}$, k, and ω, and its magnitude in the form $B = B_{\rm max}\cos(kx - \omega t)$.

<u>Solution</u> *(i)* During one oscillation (one period) the wave moves by a distance equal to the wavelength of the wave. Therefore $T = \lambda/c$. If one oscillation take time T, in one second the number of oscillations will be $f = T^{-1} = c/\lambda = \frac{3\times 10^8 \text{m/s}}{50\ \text{m}} = 6$ MHz. *(ii)* The magnetic flux through the vertical, differential surface indicated on Fig. 4 is proportional to the magnitude of the magnetic field. Therefore the rate at which this magnetic flux changes is related only to the rate at which the magnetic field varies. The flux is $\frac{d\Phi_B}{dt} = \frac{d}{dt}(B\,dx\,dy) = \frac{\partial B}{\partial t}\,dx\,dy$. The linear integral of the electric field vector can be related directly to the electric field strength at the location of the differential surface $\oint \vec{E}\cdot d\vec{s} = 0 + E(x+dx)dy + 0 - E(x)dy = \left[E(x) + \frac{\partial E}{\partial x}dx - E(x)\right]dy = \frac{\partial E}{\partial x}dx\,dy$. Therefore, from Faraday's law (using the general form for a sinusoidal wave), the above equation requires that $-E_{\rm max}k\sin(kx-\omega t) = \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} = -B_{\rm max}\omega\sin(kx-\omega t)$, from which

the magnitude of the magnetic field is $B_{\max} = \frac{k}{\omega}E_{\max} = \frac{2\pi}{2\pi f\lambda}E_{\max} = \frac{E_{\max}}{c} = \frac{22\ \mathrm{V/m}}{3\times10^{8}\ \mathrm{m/s}} = 73.3\ \mathrm{nT}$. Using the right-hand rule, when the y-component of the electric field is negative, the z-component of the magnetic field is negative too; hence $\vec{B} = -73.3\ \mathrm{nT}\hat{k}$. We can arrive at the same solution from Ampere-Maxwell's law. From the definition, the displacement current through the horizontal, differential surface is related to the rate of change in the electric flux $I_d = \epsilon_0\frac{d\Phi_E}{dt} = \epsilon_0\frac{d}{dt}(Edxdz) = \epsilon_0\frac{\partial E}{\partial t}dxdy$. The linear integral of the magnetic field vector can be related directly to the magnetic field strength at the location of the differential surface $\oint \vec{B}\cdot d\vec{s} = 0 - B(x+dx)dz + 0 + B(x)dz = \left[-B(x) - \frac{\partial B}{\partial x}dx + B(x)\right]dz = -\frac{\partial B}{\partial x}dxdz$. Ampere-Maxwell's law yields $-B_{\max}k\sin(kx-\omega t) = \frac{\partial B}{\partial x} = -\mu_0\epsilon_0\frac{\partial E}{\partial t} = -\mu_0\epsilon_0 E_{\max}\omega\sin(kx-\omega t)$, from which $B_{\max} = \mu_0\epsilon_0\frac{\omega}{k}E_{\max} = \frac{E_{\max}}{c}$ that leads to the same answer. *(iii)* From the frequency of the wave, its angular frequency is $\omega = 2\pi f = 2\pi\cdot 6\ \mathrm{MHz} = 37.7\ \mathrm{MHz}$. From the given wavelength, the value of the x-component of the propagation vector is $k = \frac{2\pi}{\lambda} = \frac{2\pi}{50\ \mathrm{m}} = 0.126\ \mathrm{m}^{-1}$. (The peak value of the magnetic field was found in part (ii)). Hence $B(x,t) = B_{\max}\cos(kx-\omega t)\hat{k} = 73.3\ \mathrm{nT}\cos(0.126\ \mathrm{m}^{-1}\cdot x - 37.7\ \mathrm{MHz}\cdot t)\hat{k}$.

5. Some science fiction writers have described solar sails that could propel interstellar spaceships. Imagine a giant sail on a spacecraft subjected to radiation pressure from our Sun. *(i)* Explain why this arrangement works better if the sail is highly reflective rather than highly absorptive. *(ii)* If the sail is assumed highly reflective, show that the force exerted by the sunlight on the spacecraft's sail is given by $F_{\rm rad} = \frac{P_\odot A}{2\pi r^2 c}$, where $P_\odot$ is the power output of the Sun (3.8×10^{26} W), A is the surface area of the sail, r is the distance from the Sun, and c is the speed of light. (Assume that the area of the sail is much larger than the area of the spacecraft so that all the force is due to radiation pressure on the sail, only. *(iii)* Using a reasonable value for A, compute the force on the spacecraft due to the radiation pressure and the force on the spacecraft due to the gravitational force of the Sun on the spacecraft. Does this result imply that such a system will work? Explain your answer.

Solution *(i)* If the sail is reflective then it gets twice as much of a momentum kick from the light as it does if it was absorptive. This is because the reflective sail has to reflect the light back, pushing the sail back harder. This accelerates the sail better than simply absorbing the light. *(ii)* The radiation force can be expressed in terms of the radiation pressure, $F_{\rm rad} = P_{\rm rad}A$, where A is the area of the sail. The radiation pressure is $2I/c$, where I is the intensity, and the factor of 2 comes from the fact that the sail is reflective. Now, the intensity comes from the sun, and can be written as $I = \frac{P_\odot}{4\pi r^2}$ where r is the distance to the sail. Thus, we finally find that the force is $F_{\rm rad} = \frac{P_\odot A}{2\pi r^2 c}$. *(iii)* The ratio of the radiation force to the Newtonian gravitational force is $\frac{F_{\rm rad}}{F_G} = \frac{\frac{P_\odot A}{2\pi r^2 c}}{\frac{G_N m M_\odot}{r^2}} = \frac{P_\odot A}{2\pi G_N m M_\odot c}$. All of these are constants, except for the area, A, and mass, m, of the ship. So, plugging in the numbers for everything except m and A, $\frac{F_{\rm rad}}{F_G} = \frac{P_\odot A}{2\pi G_N m M_\odot c} = \frac{3.8\times10^{26}}{2\pi\cdot 6.672\times10^{-11}\cdot 2.00\times10^{30}\cdot 3\times10^{8}}\frac{A}{m} = 0.0015\ \frac{A}{m}$. In order for this to be an effective means of propulsion we need $F_{\rm rad}/F_G > 1$, which requires that $0.0015\frac{A}{m} > 1 \to m/A < 0.0015$. So, we would need a tremendously huge sail, and a very light ship. For example, for a 1000 kg ship we would need an area bigger of at least 670,000 square meters, would be a circle of more than 460 meters! It seems like this would be a practically difficult method of space travel, at least if powered by the Sun. However, perhaps by firing lasers from the surface of the Earth to the sail and pushing it with extra light we could build up a good speed.

6. A pulsed laser fires a 1000 MW pulse that has a 200 ns duration at a small object that has a mass equal to 10.0 mg and is suspended by a fine fiber that is 4.00 cm long. If the radiation is completely absorbed by the object, what is the maximum angle of deflection of this pendulum? [*Hint*: Think of the system as a ballistic pendulum and assume the small object was hanging vertically before the radiation hit it.]

Solution Consider the pendulum in Fig. 5. Initially the object has zero energy, but it is then hit with the pulse which gives it a kick, lifting it up to a height h, which can be expressed in terms of the angle as $h = L - L\cos\theta = L(1-\cos\theta)$. When it is pushed up to the height h, the object has a potental energy $E_P = mgh = mgL(1-\cos\theta)$. Equating this to the initial kinetic energy of the pulse $E_K = mgL(1-\cos\theta)$. Solving this expression for the angle gives $\theta = \cos^{-1}\left[1 - \frac{E_K}{mgL}\right]$. Now, we just need to figure out the kinetic energy of the pulse. The pulse carries momentum, which transfers to the object. Hence, $p_{\text{pulse}} = p_{\text{object}}$, which gives it kinetic energy equal to the kinetic energy of the pulse. Hence, $E_{K,\text{pulse}} = E_{K,\text{object}}$. Now, $E_{K,\text{object}} = \frac{p^2_{\text{object}}}{2m} = \frac{p^2_{\text{pulse}}}{2m}$. Hence, $\theta = \cos^{-1}\left[1 - \frac{p^2_{\text{pulse}}}{2m^2gL}\right]$. To finish we just need to find the momentum of the pulse. This can be found by looking at the energy of the wave, which is related to the momentum by $E = pc$, and the energy can be related to the power, P , by $E = P\Delta t$, which, finally, gives $\theta = \cos^{-1}\left[1 - \frac{P^2\Delta t^2}{2m^2c^2gL}\right]$. Thus, we can plug in the numbers to find $\theta = \cos^{-1}\left[1 - \frac{P^2\Delta t^2}{2m^2c^2gL}\right] = \cos^{-1}\left[1 - \frac{(10^9)^2(2\times10^{-7})^2}{2(0.01)^2(3\times10^8)^2\cdot 9.8\cdot 0.04}\right] = 0.0061^\circ$.

7. An electromagnetic wave has a frequency of 100 MHz and is traveling in a vacuum. The magnetic field is given by $B(z,t) = 1.00\times10^{-8}\cos(kz-\omega t)\hat{\imath}$. *(i)* Find the wavelength and the direction of propagation of this wave. *(ii)* Find the electric field vector, $\vec{E}(z,t)$. *(iii)* Determine the Poynting vector, and use it to find the intensity of the wave.

Solution *(i)* The direction is easy to find by looking at the sign of the ωt term in the wave. Since it is negative, this tells us that the wave is traveling to the right. Because wave depends on z, this tells us that the wave is moving along the z direction. Furthermore, since $\lambda f = c$, where λ is the wavelength, f is the frequency, and c is the speed of light, we can solve for the wavelength, $\lambda = \frac{c}{f} = \frac{3\times10^8}{100\times10^6} = 3.00$ m. *(ii)* The electric field has the same form as the magnetic field, but it points along $-y$ (it has to be perpendicular to both the direction of the magnetic field and the direction of the propagation of the wave, such that $\vec{E}\times\vec{B}$ points along z). We also know that the amplitude of the electric field is related to the amplitude of the magnetic field by $E = cB$. Thus, $E = 1.00\times10^{-8}\cdot3\times10^8 = 3$ V/m. Hence, the electric field is $\vec{E} = -3.00\text{ V/m}\cos(kz-\omega t)\hat{\jmath}$. *(iii)* The Poynting vector is $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = -\frac{1}{\mu_0}E_0B_0\cos^2(kz-\omega t)\hat{\jmath}\times\hat{\imath} = \frac{1}{\mu_0}E_0B_0\cos^2(kz-\omega t)\hat{\jmath}\times\hat{\imath}\hat{k}$. Plugging in for the amplitudes and μ_0 gives $\vec{S} = \frac{3.00\times10^{-8}}{4\pi\times10^{-7}}\cos^2(kz-\omega t)\hat{k} = 0.024\text{ W/m}^2\cos^2(kz-\omega t)\hat{k}$. The intensity of the wave is given by the average of the Poynting vector, which gives a factor of 1/2 from the cosine term. Thus, $I = \frac{0.024}{2} = 12\text{ mW/m}^2$.

8. A dish antenna having a diameter of 20 m receives (at normal incidence) a radio signal from a distant source as shown in Fig. 6. The radio signal is a continuous sinusoidal wave with amplitude

$E_{\rm m} = 0.2\mu\text{V/m}$. Assume the antenna absorbs all the radiation that falls on the dish. *(i)* What is the amplitude of the magnetic field in this wave? *(ii)* What is the intensity of the radiation received by the antenna? *(iii)* What is the power received by the antenna? *(iv)* What force is exerted by the radio waves on the antenna?

Solution *(i)* The magnitude of the electric field vector and the magnitude of the magnetic field vector are proportional to each other $B_{\rm m} = E_{\rm m}/c = \frac{0.2\times10^{-6}\ \text{V/m}}{3\times10^{8}\text{m/s}} = 6.7\times10^{-16}$ T. *(ii)* By definition, the intensity of electromagnetic wave is equal to the average value of the magnitude of the Poynting vector. It can also be expressed in terms of the magnitude of the electric field vector or the magnitude of the magnetic field vector. The intensity is $I = \langle S\rangle = \frac{E_{\rm m}^2}{2\mu_0 c} = \frac{cB_{\rm m}^2}{2\mu_0} = \frac{3\times10^{8}\ \text{m/s}(6.7\times10^{-16}\ \text{T})^2}{2\cdot4\pi\times10^{-7}\ \text{Tm/A}} = 5.31\times10^{-17}\ \text{W/m}^2$. *(iii)* The power received by the antenna is related to the size of the antenna and the intensity of the approaching wave $\langle P\rangle = I\frac{\pi D^2}{4} = 5.31\times10^{-17}\ \frac{\text{W}}{\text{m}^2}\frac{\pi(20\text{m})^2}{4} = 1.67\times10^{-14}$ W. *(iv)* The force exerted on the antenna is equal to product of the antenas area and the wave pressure, related to the magnitude of the Poyting vector $F = PA = \frac{\langle S\rangle}{c}\frac{\pi D^2}{4} = \frac{5.31\times10^{-17}\ \text{W/m}^2}{3\times10^{8}\ \text{m/s}}\cdot\frac{\pi(20\ \text{m})^2}{4} = 5.56\times10^{-23}$ N.

9. Show that any function of the form $y(x,t) = f(x-ct)+g(x+ct)$ satisfies the one-dimensional wave equation for light, $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$.

Solution This problem relies on using the chain rule. Suppose that we call $u = x\pm ct$ (this takes care of both functions at once). Then, $\frac{\partial}{\partial x}f(x-ct) = \frac{\partial}{\partial x}f(u) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}$, while $\frac{\partial^2}{\partial x^2}f(x-ct) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} = \frac{\partial^2 f}{\partial u^2}$. Furthermore, $\frac{\partial}{\partial t}f(x-ct) = \frac{\partial}{\partial t}f(u) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \pm c\frac{\partial f}{\partial u}$, while $\frac{\partial^2}{\partial t^2}f(x-ct) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \pm c\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\right) = \pm c\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial t} = c^2\frac{\partial^2 f}{\partial u^2}$. This means that $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}f(x-ct) = \frac{1}{c^2}c^2\frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 f}{\partial u^2}$, and so $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial u^2} = 0$. Thus, we see that these functions do, indeed, satisfy the one-dimensional wave equation for light.

10. Suppose that we have a cylindrical capacitor, as seen in the Fig. 7. Suppose further that we put an AC current across the plates, starting at a low frequency, ω. As the voltage alternates, the positive charge on the top plate is take off and negative charge is put on. While that is happening, the electric field disappears and then builds up in the opposite direction. As the charge sloshes back and forth slowly, the electric field follows. At each instant the electric field is uniform, as shown in the figure, except for some edge effects which we are going to disregard. We can write the electric field as $E = E_0\cos(\omega t)$, where $E_0 = Q_0/\epsilon_0 A$ is constant, and $A = \pi a^2$ is the area of the plate. Now will this continue to be right as the frequency goes up? No, because as the electric field is going up and down, there is a flux of electric field through any circular loop, say Γ, of radius r inside the capacitor. And, as you know, a changing electric field acts to produce a magnetic field. From Maxwell's equations, the magnetic field is given by a

$$c^2\oint\vec{B}\cdot d\vec{\ell} = \frac{d}{dt}\int\vec{E}\cdot d\vec{A} \Rightarrow c^2 B 2\pi r = \frac{d}{dt}(E\pi r^2),$$

or

$$B = -\frac{\omega r}{2c^2}E_0\sin(\omega t).$$

So, the changing electric field has produced a magnetic field circulating around inside the capacitor, and oscillating at the same frequency as the electric field. Now, are we done? No! This magnetic field also oscillates, which produces a new electric field! The uniform field, $E_1 = E_0 \cos(\omega t)$, is only the first term! The changing magnetic field produces a new electric field, E_2, such that the total field is $E = E_1 + E_2$. Now, in general, E_2 is also oscillating! This means that there will be a new magnetic field from E_2, which will be oscillating, which will create a new electric field, E_3, which will create a new magnetic field.... Your task is to calculate the first four terms of the series, enough to get the pattern, and write the total electric field, taking the field at the center of the capacitor to be exactly $E_0 \cos(\omega t)$, (i.e., there is no correction at the center). Then, compare your result with Bessel functions, and see if you can write the full electric field you find in terms of one of the Bessel functions, in closed form. Can you find an exact expression in terms of one of the Bessel functions for the magnetic field?

Solution As given above, the uniform field generates a magnetic field, $B = -\frac{\omega r}{2c^2} E_0 \sin(\omega t)$. Now, Faraday's law reads $\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$ Now, we want to take a loop for which the electric field is constant everywhere along the integration path. We'll take a rectangular loop, that goes up along the axis of the capacitor, out to a radial distance r along the top plate, down vertically to the bottom plate, and then back to the axis. Now, the field is $E = E_1 + E_2$, but the loop integral of E_1 is zero, since E_1 is uniform. Therefore, only E_2 contributes. Now along the loop, $E_2 = 0$ at the center, as per our assumption, while the part of the loop running along the plates is zero since the field is perpendicular to the path. So, the whole integral is just $-E_2(r)h$, where h is the distance between the plates. The negative sign comes in because the path travels down while the field is pointing up. Now, the flux of B through the surface bounded by the loop is just $\Phi_B = \int B dA = h \int_0^r B dr$, since the loop is a rectangle of height h, and we have to integrate over the width of the rectangle, since B changes with distance. Now, using our expression for $B = -\frac{\omega r}{2c^2}\ E_0 \sin(\omega t)$, then $h \int_0^r B dr = -h \frac{\omega r^2}{4c^2} E_0 \sin(\omega t)$, and so, from Faraday's law, $E_2(r) = -\frac{\omega^2 r^2}{4c^2} E0 \cos(\omega t)$. So, we have so far that $E = E_1 + E_2$, or $E = \left(1 - \frac{\omega^2 r^2}{2^2 c^2}\right) E_0 \cos(\omega t)$. Now, we need to continue on. This new term, E_2 will produce a new magnetic field. Let's call the magnetic field that we found before B_1. Then E_2 produces a new magnetic field B_2 such that the total field is $B = B_1 + B_2$. To get B_2, we apply the same trick that we used to find B_1, that is $c^2 \oint \vec{B}_2 \cdot d\vec{s} = \frac{d}{dt} \int \vec{E}_2 \cdot d\vec{A}$. Taking the same loop gives $c^2 B_2(2\pi r)$ for the left hand side. Now, because E_2 varies with radius, the right hand integral reads $\int E dA = 2\pi \int_0^r E_2 r dr = -2\pi \frac{\omega^2}{4c^2} E_0 \cos(\omega t) \int_0^r r^3 dr = -2\pi \frac{\omega^2 r^4}{4^2 c^2} E_0 \cos(\omega t)$. Thus, taking the derivative gives $B_2 = \frac{\omega^3 r^3}{2^4 c^4} E_0 \sin(\omega t)$. But, we need to keep going! This changing magnetic field produces a new electric field, E_3, which we can calculate just as before for E_2. Doing so gives $E_3 = \frac{\omega^4 r^4}{2^2 4^2 c^4} E_0 \cos(\omega t)$. This, again, produces a new magnetic field, which produces a new electric field... The pattern keeps continuing, and we keep iterating. The next correction to the electric field is $E_4 = -\frac{1}{2^2 4^2 6^2} \left(\frac{\omega r}{c}\right)^6 E_0 \cos(\omega t)$. So, the electric field is given by $E = E_0 \left[1 - \frac{1}{(1!)^2}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega r}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega r}{2c}\right)^6 + \cdots\right] \cos(\omega t)$. Now, we can look up the Bessel functions, and we find that the zeroth-order function, $J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$, and so, we finally find that the electric field is given by $E = E_0 J_0\left(\frac{\omega r}{c}\right) \cos(\omega t)$. We can look up the series for the magnetic field to find the first-order Bessel function, $J_1(x) = \left(\frac{x}{2}\right) - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \cdots$, to find $B = -\frac{E_0}{c} J_1\left(\frac{\omega r}{c}\right) \sin(\omega t)$. (We could also plug the electric field back into the Maxwell equations, noting that $\int J_1(x) dx = -J_0(x)$.) That completely solves

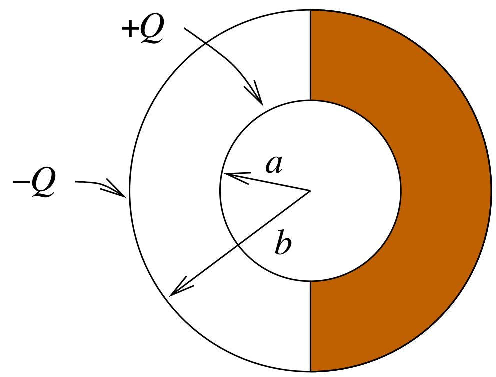

Figure 1: Problem 1.

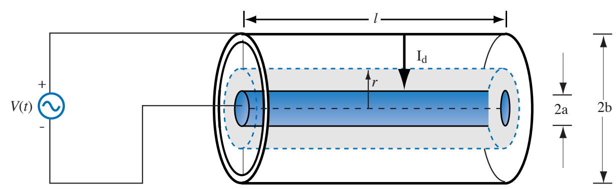

Figure 2: Problem 2.

the problem!

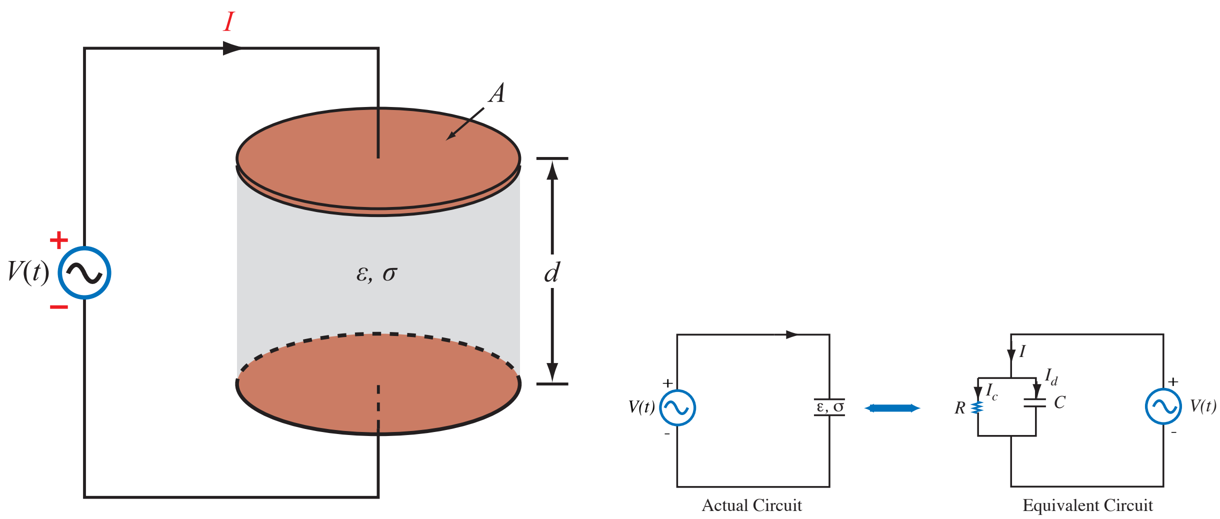

Figure 3: Problem 3.

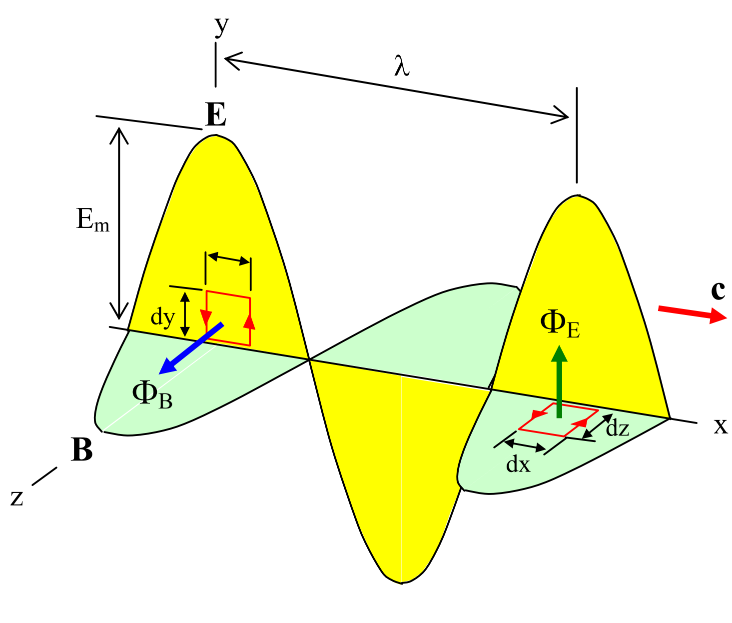

Figure 4: Problem 4.

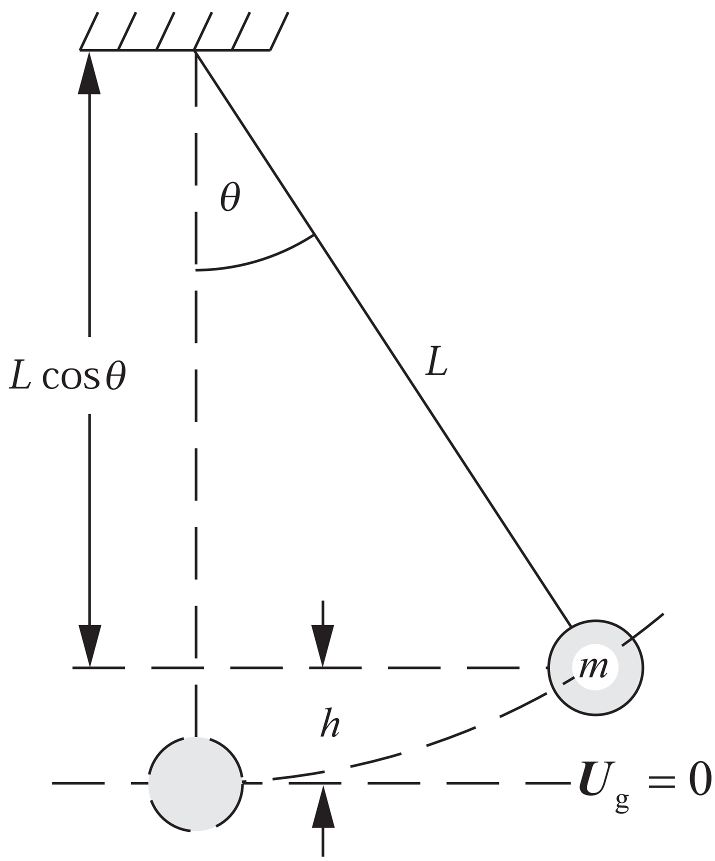

Figure 5: Problem 6.

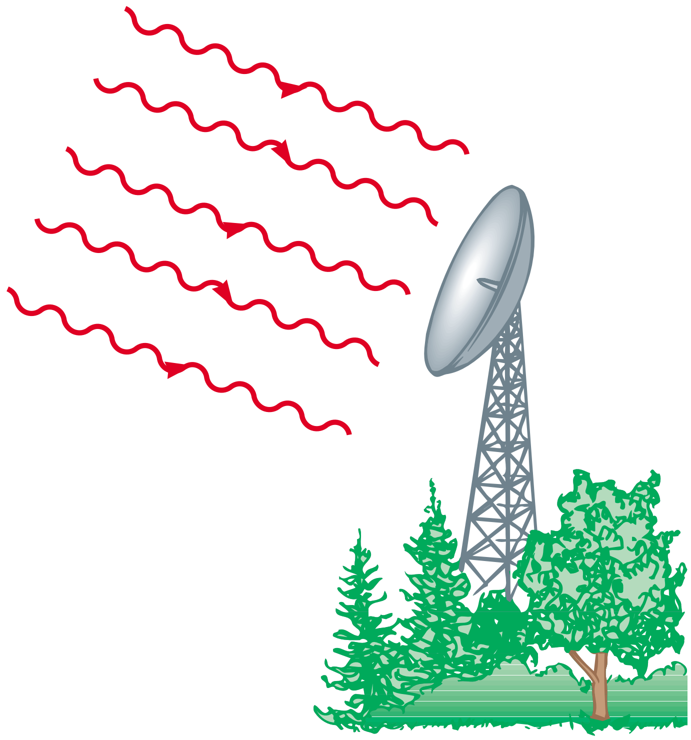

Figure 6: Problem 8.

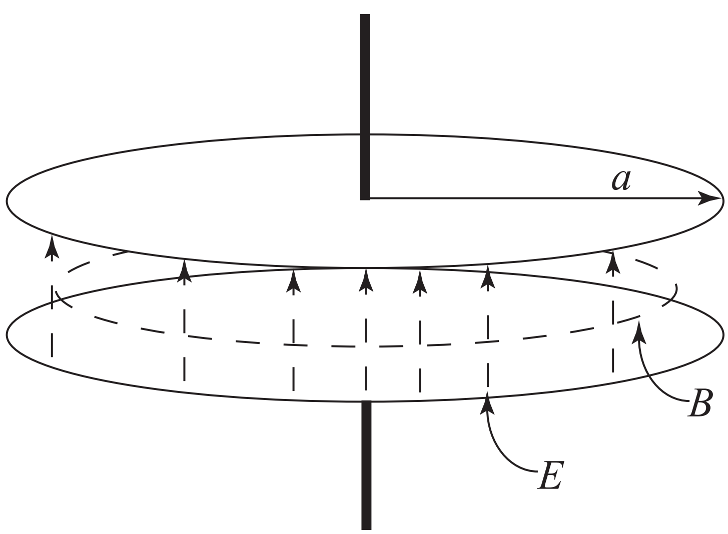

Figure 7: Problem 10.