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Problems set # 11 **Physics 169** May 5, 2015

1. A semicircular conductor of radius $R = 0.250$ m is rotated about the axis AC at a constant rate of 120 rev/min (Fig. 1). A uniform magnetic field in all of the lower half of the figure is directed out of the plane of rotation and has a magnitude of 1.30 T. *(i)* Calculate the maximum value of the emf induced in the conductor. *(ii)* What is the value of the average induced emf for each complete rotation? *(iii)* How would the answers to (i) and (ii) change if B were allowed to extend a distance R above the axis of rotation? Sketch the emf versus time *(iv)* when the field is as drawn in Fig. 1 and *(v)* when the field is extended as described in (iii).

Solution *(i)* $\varepsilon_{\rm max} = BA\omega = B\pi R^2\omega/2 = 1.30~{\rm T}\cdot\frac{\pi}{2}\cdot(0.25~{\rm m})^2\cdot 4.00\pi~{\rm rad/s} = 1.60~{\rm V}$. *(ii)* $\bar{\varepsilon} = \frac{1}{2\pi}\int_0^{2\pi}\varepsilon~d\theta = \frac{BA\omega}{2\pi}\int_0^{2\pi}\sin\theta~d\theta = 0$. *(iii)* The maximum and average ε would remain unchanged. *(iv)* See Fig. 2. *(v)* See Fig. 2.

2. An AC power supply produces a maximum voltage of $V_0 = 100$ V. This power supply is connected to a $24.0 - \Omega$ resistor, and the current and resistor voltage are measured with an ideal AC ammeter and an ideal AC voltmeter, as shown in Fig. 3 What does each meter read? Recall that an ideal ammeter has zero resistance and an ideal voltmeter has infinite resistance.

Solution The meters measure the rms values of potential difference and current. These are $V_{\rm rms} = \frac{V_0}{\sqrt{2}} = 70.7$ V and $I_{\rm rms} = \frac{V_{\rm rms}}{R} = 2.95$ A.

3. *(i)* For the series RLC connection of Fig. 4, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors. *(ii)* An RLC circuit consists of a 150-Ω resistor, a 21-μF capacitor and a 460-mH inductor, connected in series with a 120-V, 60-Hz power supply. What is the phase angle between the current and the applied voltage? *(iii)* Which reaches its maximum earlier, the current or the voltage?

Solution *(i)* For the series connection, the instantaneous voltage across the system is equal to the sum of voltage across each element. The phase angle between the voltage (across the system) and the current (through the system) is: $\phi = \arctan\frac{X_L - X_C}{R}$. In Fig. 5 the phasor diagram for a series RLC circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$. On the one hand, in the inductive case, $V_{0,L} > V_{0,c}$, we see that $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitance case, $V_{0,C} > V_{0,L}$, we have that $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ. *(ii)* From the definition, the inductive reactance of the inductor is $X_L = \omega L = 2\pi\cdot 60~{\rm Hz}\cdot 0.46~{\rm H} = 173~\Omega$. From the definition, the capacitive reactance of the capacitor is $X_C = \frac{1}{\omega C} = \frac{1}{2\pi\cdot 60~{\rm Hz}\cdot 21\times 10^{-6}~{\rm F}} = 126~\Omega$. The phase angle between the voltage (across the system) and the current (through the system) is: $\phi = \arctan\frac{X_L - X_C}{R} = \arctan\frac{174~\Omega - 126~\Omega}{150~\Omega} = 17.4^\circ$ *(iii)* The voltage leads the current.

4. Figure 6 shows a parallel RLC circuit. The instantaneous voltage (and rms voltage) across each of the three circuit elements is the same, and each is in phase with the current through the

resistor. The currents in the capacitor and the inductor lead or lag behind the current in the resistor. *(i)* Show that the rms current delivered by the source is $I_{\rm rms} = V_{\rm rms}\left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{1/2}$. *(ii)* Show that the phase angle between the voltage and the current is $\tan\varphi = -R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)$. *(iii)* Draw a phasor diagram for the currents. The amplitudes of the currents across all the circuit elements involved should be represented with phasors.

Solution We are asked to find the impedance and the phase angle for this system of elements connected in parallel. It will be easier to analyze the complex impedance. For elements connected in parallel the equivalent complex impedance of the system is equal to the sum of inverse impedance of each element $\frac{1}{Z_{\rm eq}} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C = \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)$, or $Z_{\rm eq} = \frac{1}{\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)} = \frac{\frac{1}{R} - i\left(\omega C - \frac{1}{\omega L}\right)}{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$. *(i)* Hence $I_{\rm rms} = \frac{I_0}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left|\frac{V_0}{Z_{\rm eq}}\right| = \frac{V_0}{\sqrt{2}}\left|\frac{1}{Z_{\rm eq}}\right| = V_{\rm rms}\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$, and *(ii)* $\tan\varphi = \frac{{\rm Im} Z_{\rm eq}}{{\rm Re} Z_{\rm eq}} = \frac{-\left(\omega C - \frac{1}{\omega L}\right)}{\frac{1}{R}}$. *(iii)* The phasor diagram for a parallel RLC circuit is shown in Fig. 7, for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

5. Draw to scale a phasor diagram showing Z, X_L, X_C, and φ for an AC series circuit for which $R = 300\ \Omega$, $C = 11\ \mu$F, $L = 0.2$ H, and $f = 500/\pi$ Hz.

Solution Before drawing the diagram, we find the complex impedance of each element. The complex impedance of a resistor is equal to its resistance, $Z_R = R = 300\ \Omega$. It is therefore a real number which we marked in the complex plane on the Re axis of Fig. 8. The complex impedance of an inductor is an imaginary number dependent on the angular frequency of the current and the inductance of the inductor $Z_L = i\omega L = 2\pi \cdot \frac{500}{\pi}\ {\rm s}^{-1} \cdot 0.2\ {\rm H}\cdot = 200i\ \Omega$. This number is on the Im axis. Finally, the complex impedance of the capacitor is also an imaginary number dependent on the angular frequency of the current and the capacitance of the capacitor $Z_C = -\frac{i}{\omega C} = -\frac{i}{2\pi}\frac{\pi}{500}\ {\rm s}^{-1}\ (11\times 10^{-6}\ {\rm F})^{-1} = -90.9i\ \Omega$. This number is also on the Im axis. Since the elements are connected in series the equivalent complex impedance is equal to the sum of the complex impedance of each element: $Z = Z_R + Z_L + Z_C = 300\ \Omega + 200i\ \Omega - 90.9i\ \Omega = (300 + 109.1i)\ \Omega$. The complex impedance of each element is indicated in Fig. 8. One tic corresponds to 100 Ω. The diagram also illustrates how to find the equivalent complex impedance. Note that the phase of the complex impedance of the resistor is 0°, the phase of the complex impedance of the capacitor is −90° and the phase of the complex impedance of the inductor is 90°.

6. Draw to scale a phasor diagram showing the relationship between the current (common for all elements) and the voltages in an AC series circuit for which $R = 300\ \Omega$, $C = 11\ \mu$F, $L = 0.2$ H, $f = 500/\pi$ Hz, and $I_0 = 20$ mA.

Solution The complex voltage across the resistor can be found by multiplying the complex current (through the resistor) by the complex impedance of the resistor $V_R(t) = I(t)Z_R$. Consistent with the given values, the absolute value of this voltage is $V_{0,R} = I_0 R = 20\ {\rm mA} \cdot 300\ \Omega = 6$ V. Since the phase of the complex impedance of the resistor is zero, the phase of the voltage across the resistor agrees with the phase of the current. The complex voltage across the inductor can be found by multiplying the complex current (through the inductor) by the complex impedance of the inductor $V_L(t) = I(t)Z_L$. Consistent with the given values, the absolute value of this voltage is $V_{0,L} = I_0X_L = 20$ mA $\cdot$ 200 $\omega = 4$ V. Since the phase of the complex impedance of the inductor is 90°, the voltage across the inductor leads with the current by a 90° phase angle. Finally, the complex voltage across the capacitor can be found by multiplying the complex current (through the capacitor) by the complex impedance of the capacitor $V_C(t) = I(t)Z_C$. Consistent with the given values, the absolute value of this voltage is $V_{0,C} = I_0 \cdot X_C = 20\text{ mA} \cdot 91\ \Omega = 1.8$ V. Since the phase of the complex impedance of the capacitor is $-90°$, the voltage across the capacitor lags behind the current by 90°. If we want to find the complex voltage across the entire system, we can add the complex voltages across all three elements (they are connected in series). I indicated this operation on the phasor diagram. That voltage can also be found by multiplying the current by the equivalent impedance of the system. Notice that the phase difference between the voltage across the system and the current through the system is equal to the phase angle of the system (the phase of the equivalent impedance). The absolute value of the voltage is $V_0 = I_0Z = 20\text{ mA}\cdot\sqrt{(300\ \Omega)^2 + (109\ \Omega)^2} \approx 6.4$ V.

7. A coil of inductance 0.12 H and resistance 3 kΩ is connected in parallel with a 0.02 μF capacitor and is supplied at 40 V at a frequency of 5 kHz; see Fig. 10. Determine *(i)* the current in the coil, and *(ii)* the current in the capacitor. *(iii)* Draw to scale the phasor diagram and measure the supply current and its phase angle; check the answer by calculation. Determine *(iv)* the circuit impedance and *(v)* the power consumed.

Solution *(i)* Inductive reactance,$X_L = 2\pi fL = 2\pi \times 5000 \times 0.12 = 3770\Omega$. Impedance of coil, $Z_1 = \sqrt{R^2 + X_L^2} = \sqrt{3000^2 + 3770^2} = 4818\ \Omega$. Current in the coil, $I_{LR} = \frac{V}{Z_1} = \frac{40}{4818} = 8.30$ mA. Branch phase angle $\phi = \arctan\frac{X_L}{R} = \arctan\frac{3770}{3000} = 51.5°$ lagging. *(ii)* Capacitive reactance, $X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi 5000\times 0.02\times 10^{-6}} = 1592\ \Omega$. Capacitor current, $I_C = \frac{V}{X_C} = \frac{40}{1592} = 25.13$ mA leading V by 90°. *(iii)* Currents I_{LR} and I_C are shown in the phasor diagram of Fig. 11. The parallelogram is completed as shown and the supply current is given by the diagonal of the parallelogram. The current I is measured as 19.3 mA leading voltage V by 74.5°. By calculation, $I\sqrt{(I_{LR}\cos 51.5°)^2 + (I_C - I_{LR}\sin 51.5°)^2} = 19.34$ mA and $\phi = \arctan\left(\frac{I_C - I_{LR}\sin 51.1°}{I_{LR}\cos 51.5°}\right) = 74.5°$. *(iv)* Circuit impedance, $Z = \frac{V}{I} = \frac{40}{19.34\times 10^{-3}} = 2.068$ kΩ. *(v)* Power consumed, $P = VI\cos\phi = 40\times 19.34\times 10^{-3}\times\cos 74.5° = 206.7$ mW. Alternatively, $P = I_R^2R = I_{LR}^2R = (8.3\times 10^{-3})^2\times 3000 = 206.7$ mW.

8. A transmission line that has a resistance per unit length of $4.5\times 10^{-4}\ \Omega$/m is to be used to transmit 5 MW over 400 miles (6.44×10^5 m); see Fig. 12. The output voltage of the generator is 4.5 kV. *(i)* What is the line loss if a transformer is used to step up the voltage to 500 kV? *(ii)* What fraction of the input power is lost to the line under these circumstances? *(iii)* What difficulties would be encountered in attempting to transmit the 5 MW at the generator voltage of 4.5 kV.

Solution *(i)* In order to send out power of $P = 5$ MW at 500 kV potential difference, the current in the grid (transmission lines) must be $I_{\rm rms,a} = \frac{P}{V_{\rm rms,a}} = 10$ A. The resistance of the two wires of the transmission lines (connected is series) is $R = 2L\lambda = 580\ \Omega$. Hence the loss of power in the lines

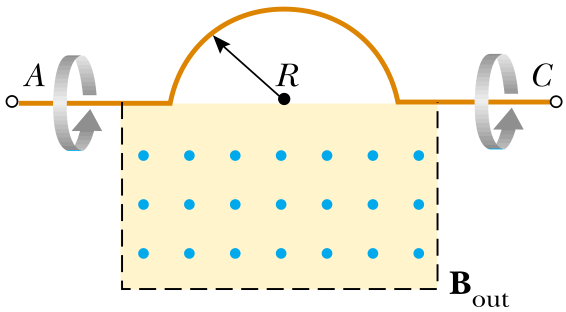

Figure 1: Problem 1.

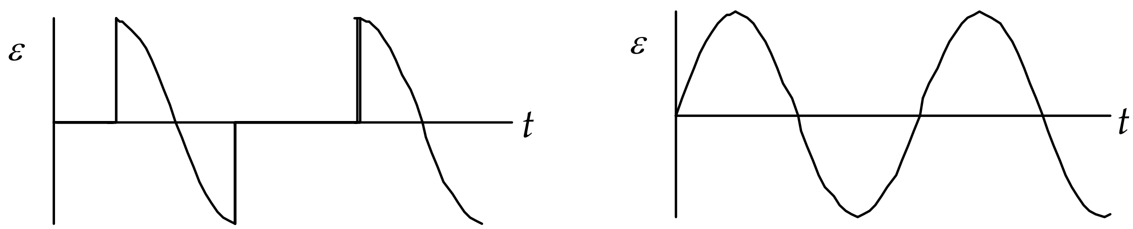

Figure 2: Solution of problem 1.

(dissipated in the lines) is $P_{\rm a} = I_{\rm rms}^2 R = (10\ \mathrm{A})^2 580\ \Omega = 58\ \mathrm{kW}$. *(ii)* A relative small fraction is lost in the lines $\frac{P_{\rm a}}{P} = \frac{58\ \mathrm{kW}}{5\ \mathrm{MW}} = 0.01 = 1\%$. *(iii)* If the power were send at 4.5 kV potential difference, the current in the grid (transmission lines) should be much larger $I_{\rm rms,c} = \frac{P}{V_{\rm rms,c}} = \frac{5\ \mathrm{MW}}{4.5\ \mathrm{kV}} = 1.1\ \mathrm{kA}$. But even if nothing were connected at the end of the lines, with the potential difference of 4.5 kV, the current in the lines would be $I_{\rm rms,max} = \frac{V_{\rm rms,c}}{R} = \frac{4.5\ \mathrm{kV}}{58\ \Omega} = 77\ \mathrm{A}$. It would be impossible to send 5 MW of power into the lines.

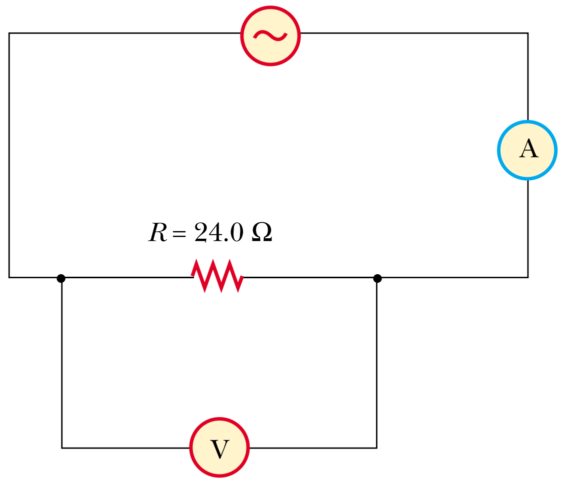

Figure 3: Problem 2.

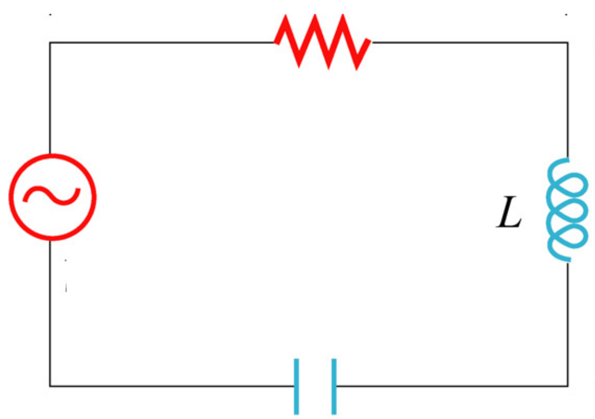

Figure 4: Problem 3.

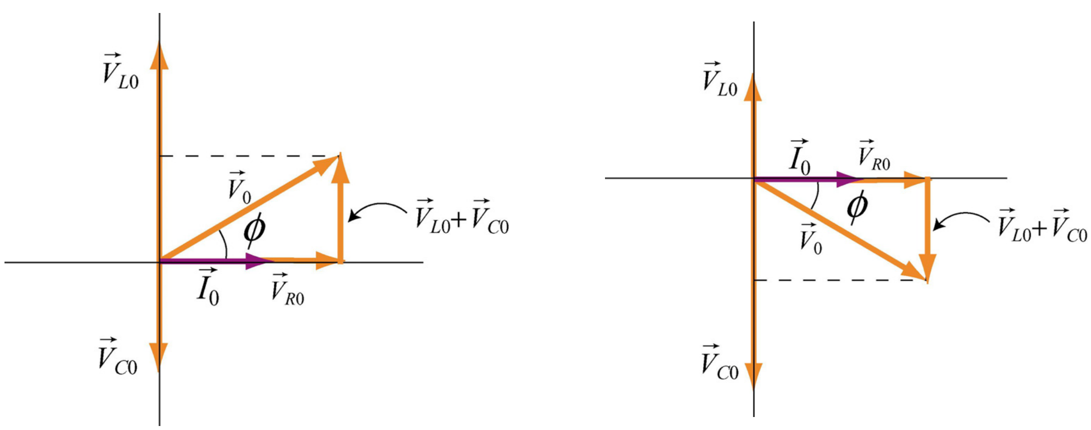

Figure 5: Phasor diagram for the series RLC circuit for $X_L > X_C$ (left) and $X_L < X_C$ (right).

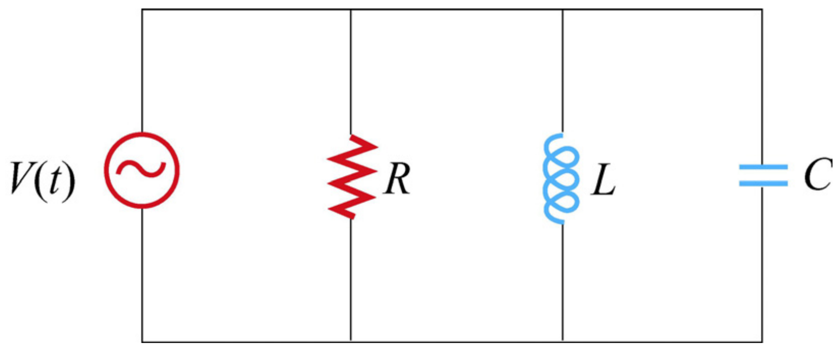

Figure 6: Problem 4.

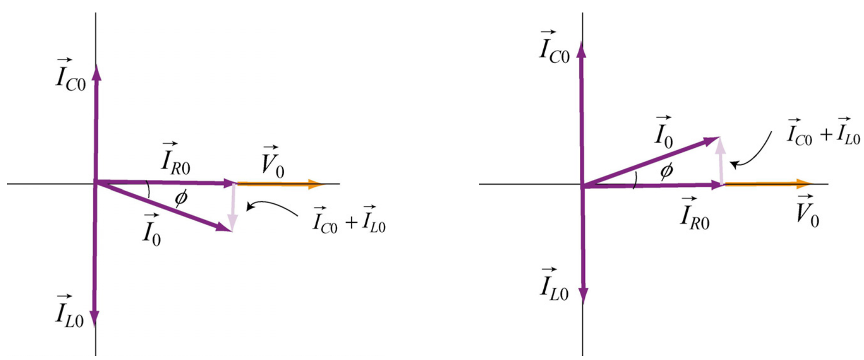

Figure 7: Phasor diagram for the parallel RLC circuit for $X_L > X_C$ (left) and $X_L < X_C$ (right).

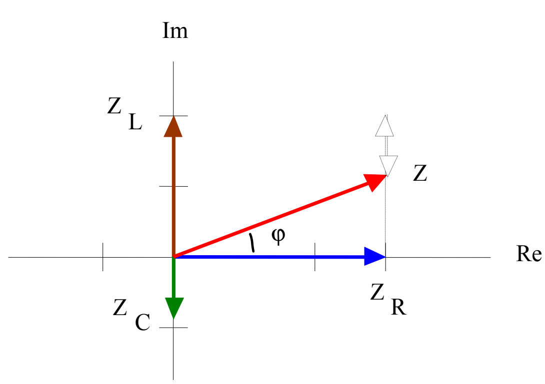

Figure 8: Problem 5.

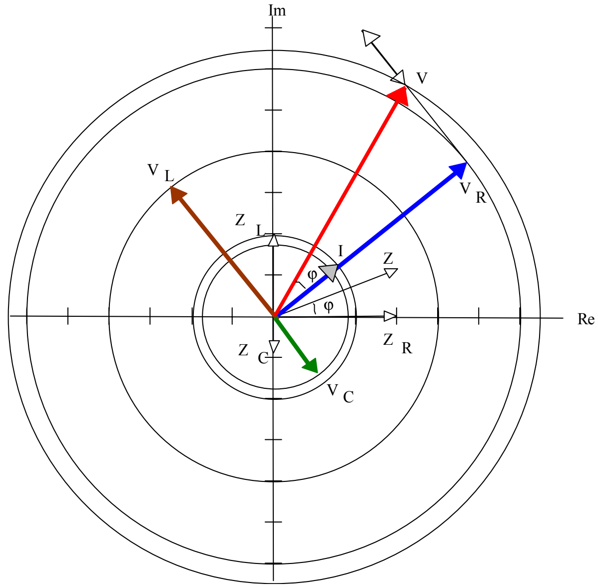

Figure 9: Problem 6.

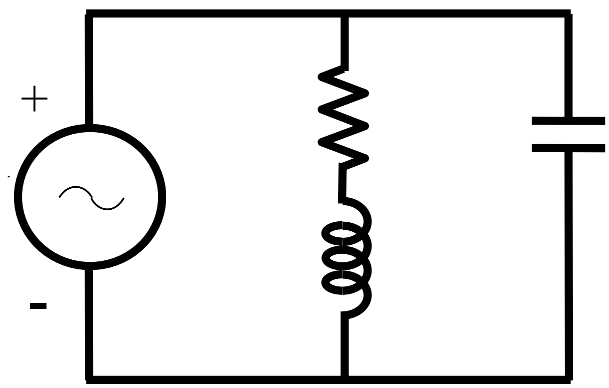

Figure 10: Problem 7.

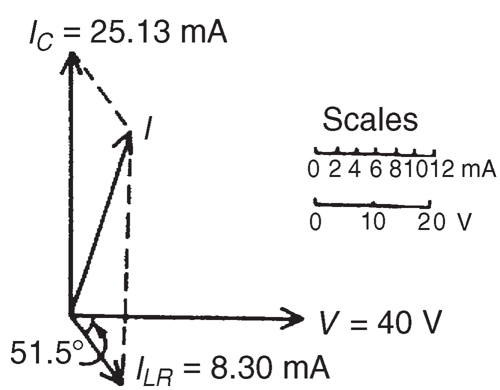

Figure 11: Solution of problem 7.

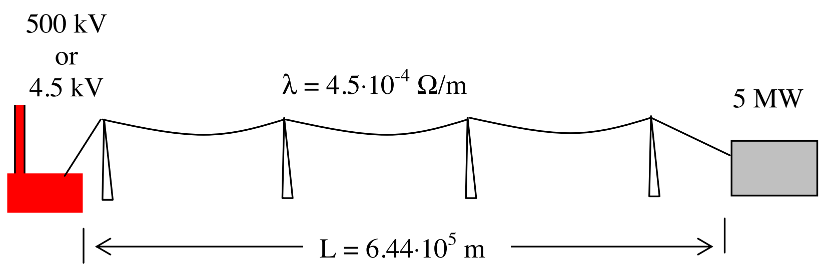

Figure 12: Problem 8.