# LESSON 9


PHYSICS 168

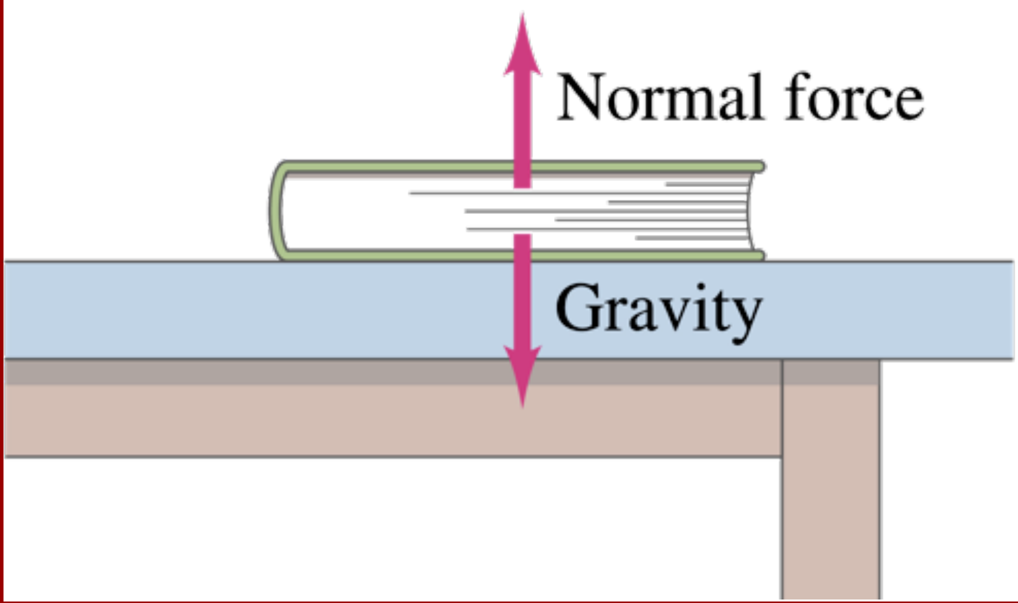
LUIS ANCHORDOQUI

# Static Equilibrium

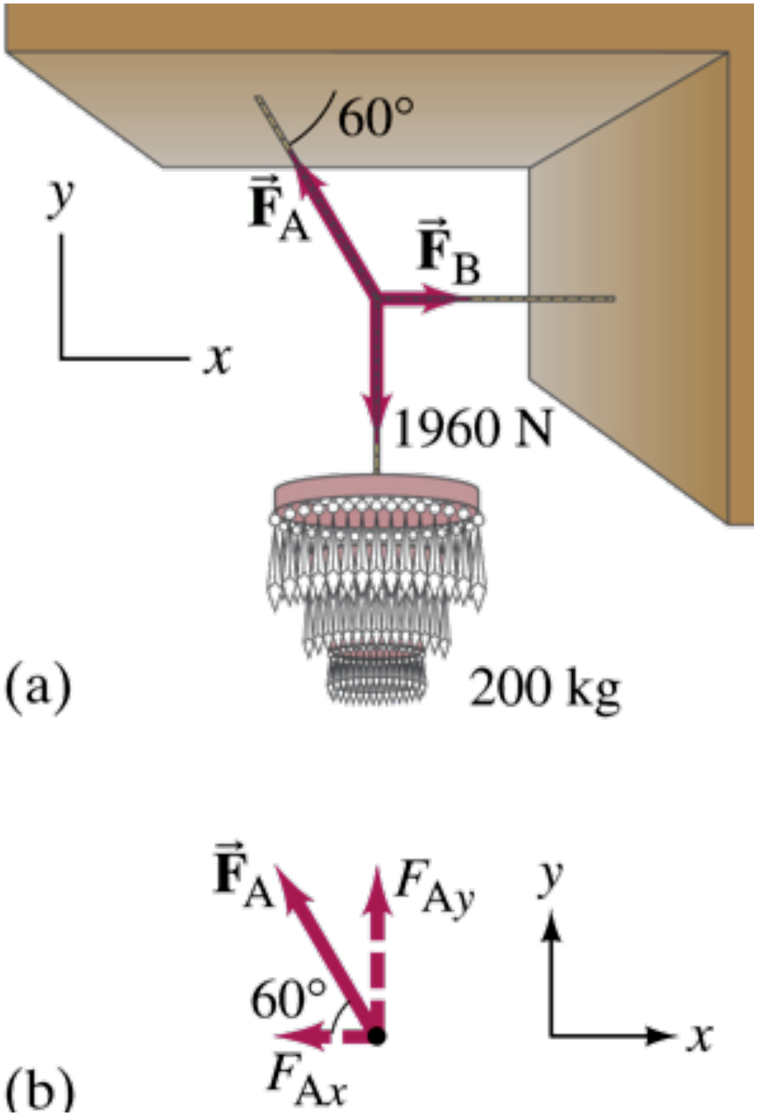
**Luis Anchordoqui**

# Conditions for Equilibrium

An object with forces acting on it but that is not moving is said to be in equilibrium



# Conditions for Equilibrium (cont'd)




**First condition for equilibrium is that forces along each coordinate axis add to zero**

**e.g.**

$$\sum F_y = F_A \sin 60^\circ - 200\, kg\, g = 0$$

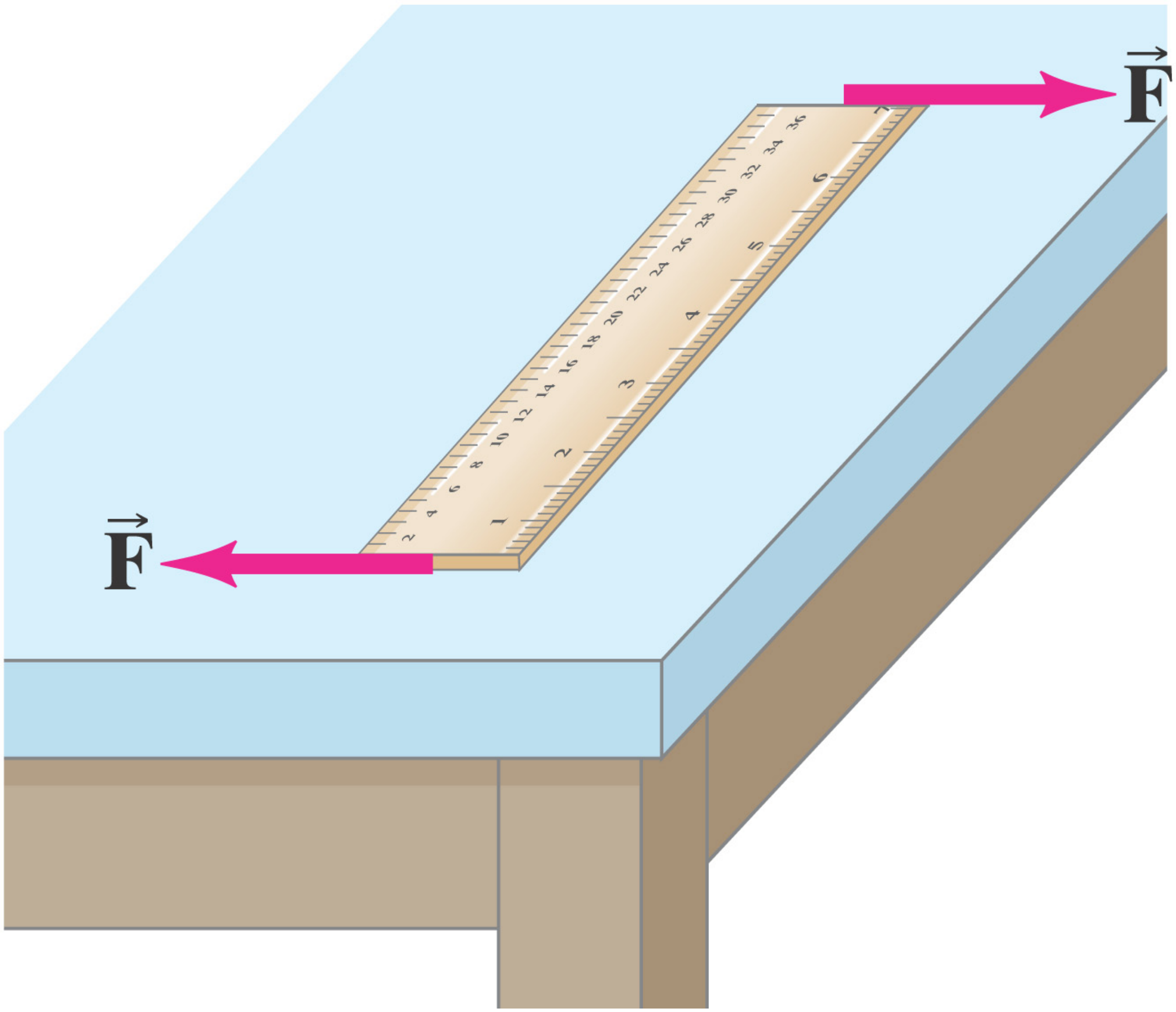
$$F_A = 2260 \text{ N}$$

$$\sum F_x = F_B - F_A \cos 60^\circ = 0$$

$$F_B = 1130 \text{ N}$$

# Conditions for Equilibrium (cont'd)

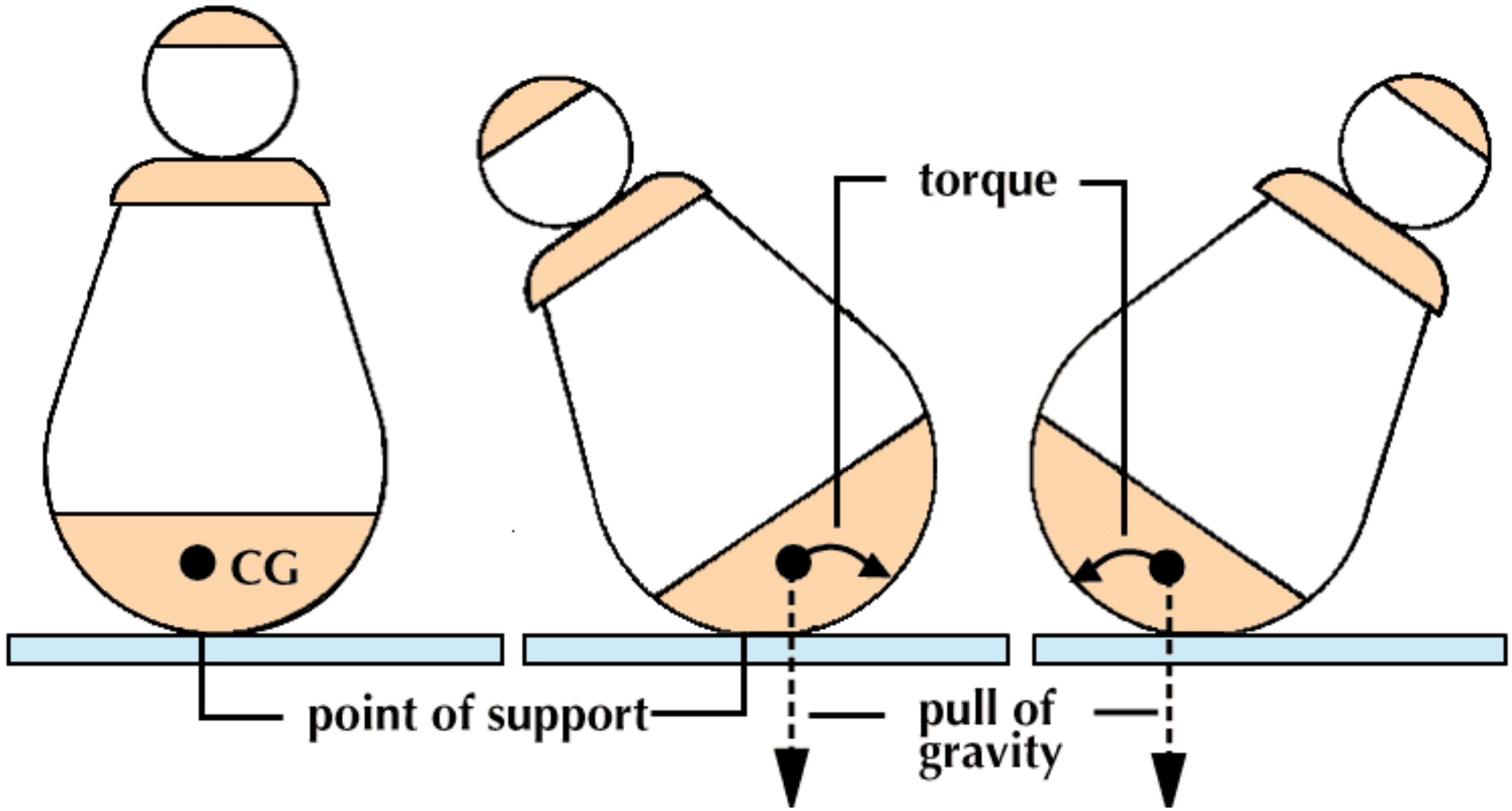
Second condition of equilibrium is that there be no torque around any axis




Choice of axis is arbitrary

# Center of Gravity

Point in a body around which the resultant torque due to **gravity** forces vanish




**center of gravity** average location of the weight of an object

# Center of Mass vs. Center of Gravity

In symmetrical objects, the center of gravity and the center of mass correspond with one another

In asymmetrical objects, the gravitation is not uniform
Thus, the center of gravity and the center of mass do not correspond with one another

The center of mass may not lie in the object's body

RECALL


The center of gravity is always present in the object's body

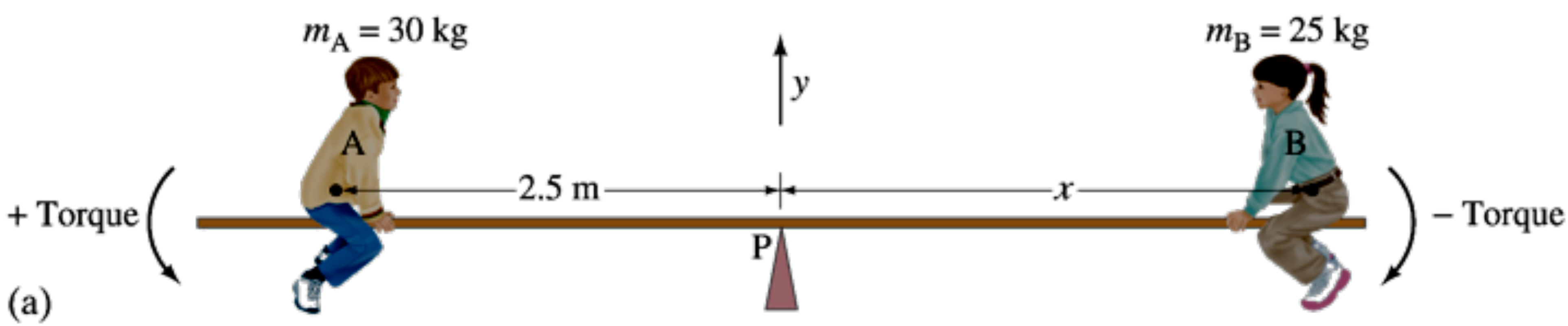
# Balancing a seesaw

A board of mass M = 2 kg serves as a seesaw for two children

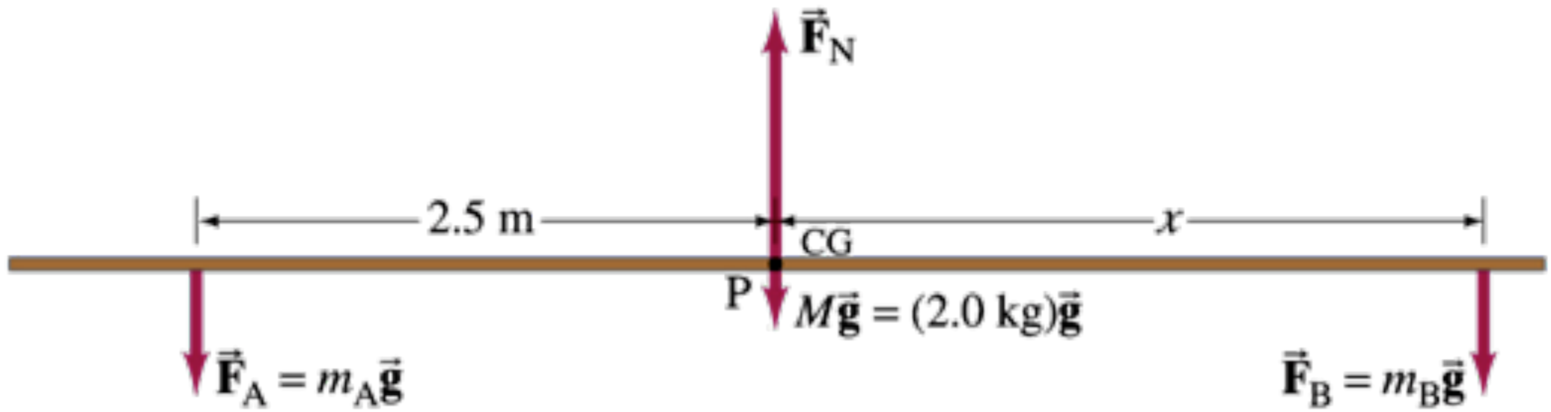
Child A h as a mass of 30 kg and sits 2.5 m from pivot point P (his CG is 2.5 m from the pivot)

At what distance x from pivot must child B of mass 25 kg, place herself to balance the seesaw?

Assume board is uniform and centered over pivot



# Balancing a seesaw (cont'd)




$M\vec{g}$ **and** $F_N$ **make no torque with respect to rotation point**

$$\sum \tau = m_A\ g\ 2.5\ \mathrm{m} - m_B g\ x = 0 \Rightarrow x = 3\ \mathrm{m}$$

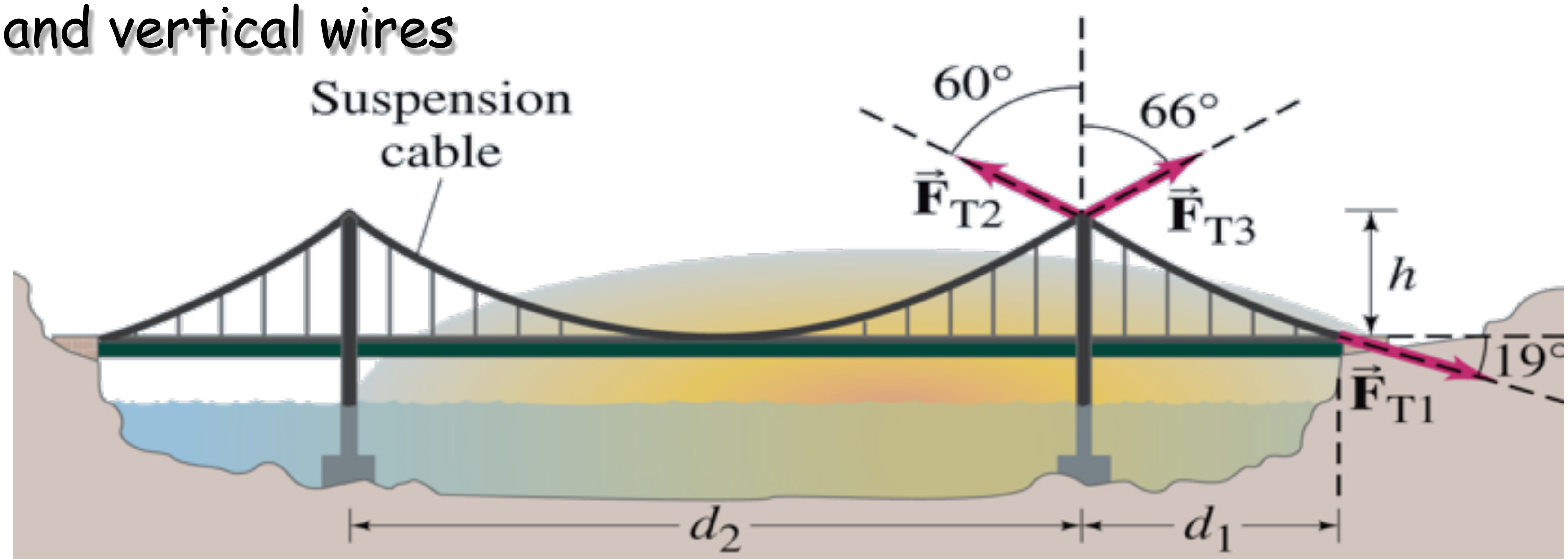
Consider right-hand (northernmost) section of Golden Gate bridge which has a length $d_1 = 343\, m$

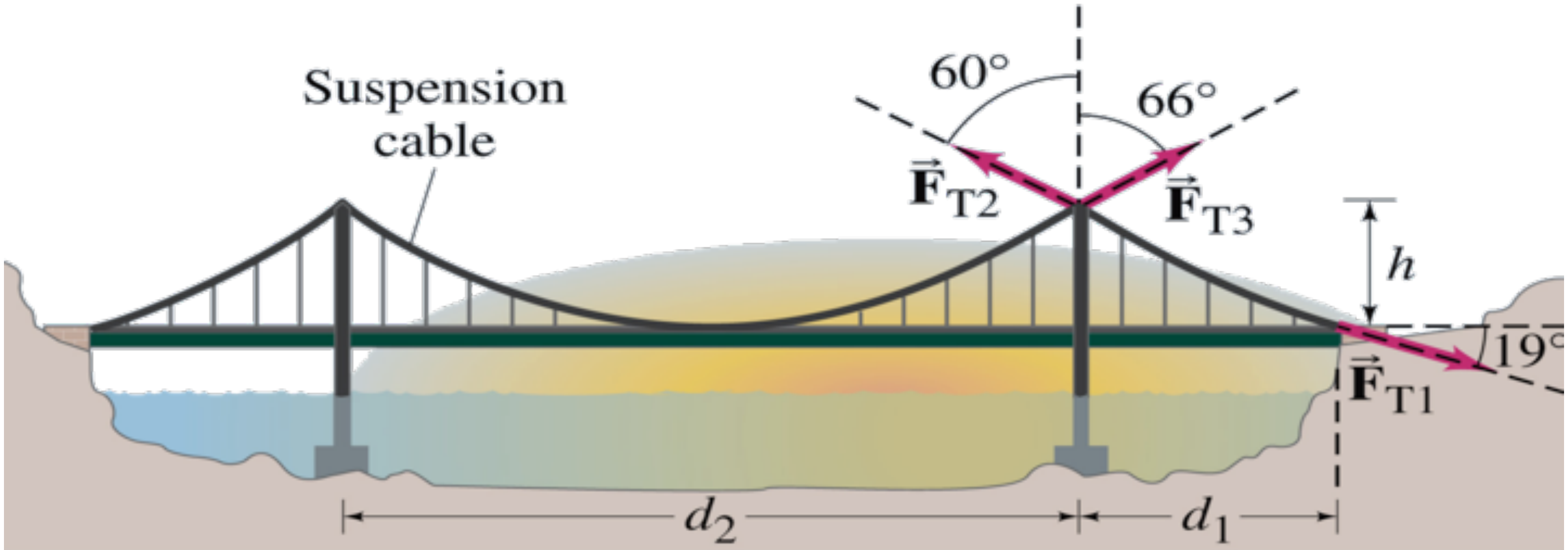
Assume CG of this span halfway between tower and achor

ⓐ Determine $F_{T1}$and $F_{T2}$ (which act on the norther most cable) in terms of $mg$ (weight of northernmost span)

ⓑ calculate height $h$ needed for equilibrium

Assume roadway is supported only by suspension cables, and neglect mass of cables and vertical wires






$$\sum F_x = F_{T_1} \cos 19^\circ - F_{T_2} \sin 60^\circ = 0$$

$$\sum F_y = F_{T_2} \cos 60^\circ - F_{T_1} \sin 19^\circ - mg = 0$$

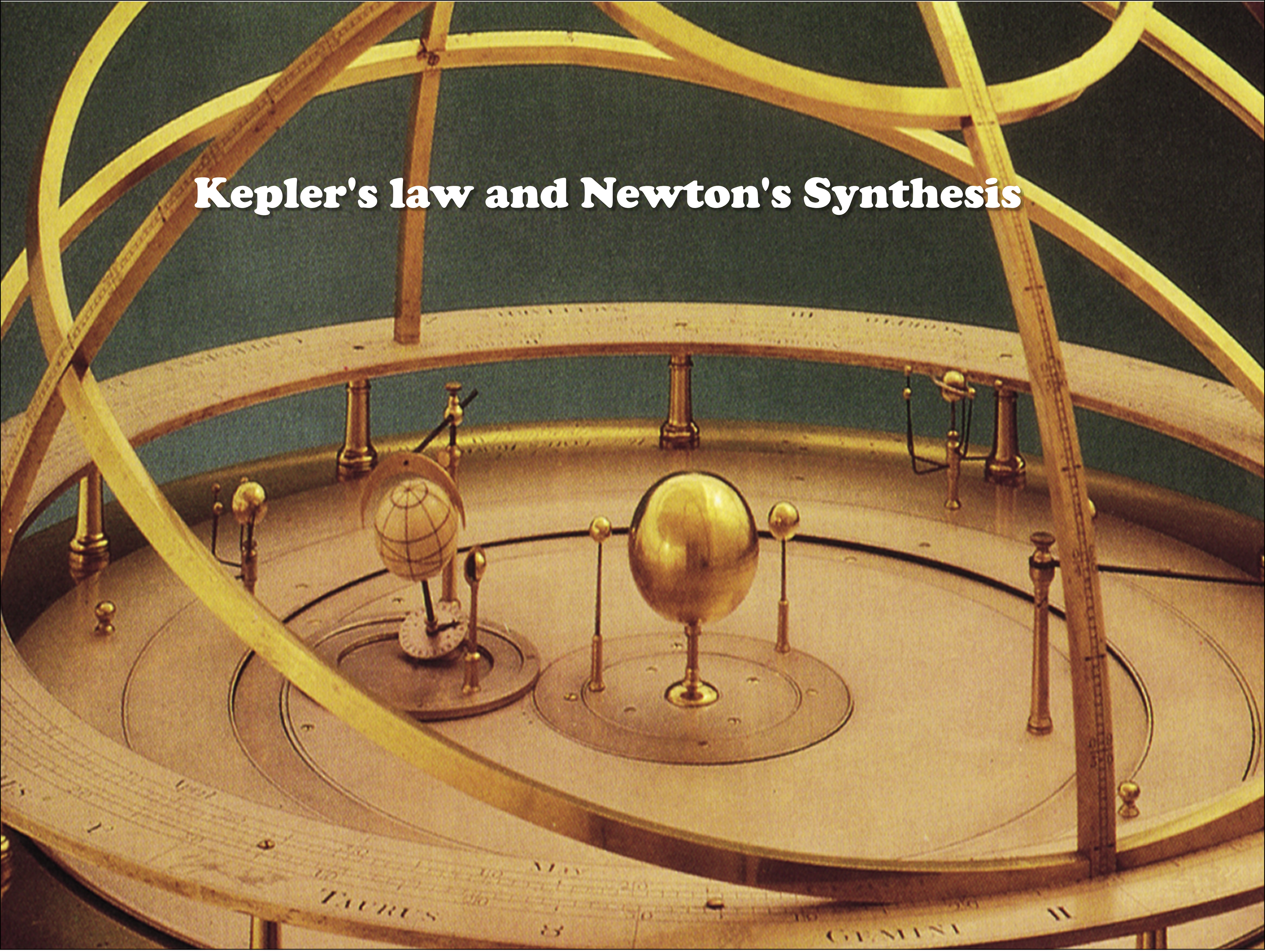
$$F_{\mathrm{T}_2} = F_{\mathrm{T}_1} \frac{\cos 19^\circ}{\sin 60^\circ}$$

$$F_{\mathrm{T}_1} = 4.5\ mg$$

$$F_{\mathrm{T}_2} = 5.0\ mg$$

$$\sum \tau = mg\frac{1}{2}d_1 + F_{T_2,x}h - F_{T_2,y}d_1 = 0 \Rightarrow h = 158\ \mathrm{m}$$

# Kepler's law and Newton's Synthesis



# Kepler's law and Newton's Synthesis

Nighttime sky with its myriad stars and shinning planets

has always fascinated people on Earth

Towards end of XVI century astronomer Tycho Brahe

studied motions of planets and made accurate observations

Using Brahe's data Johannes Kepler worked out

a detailed description of motion of planets about Sun

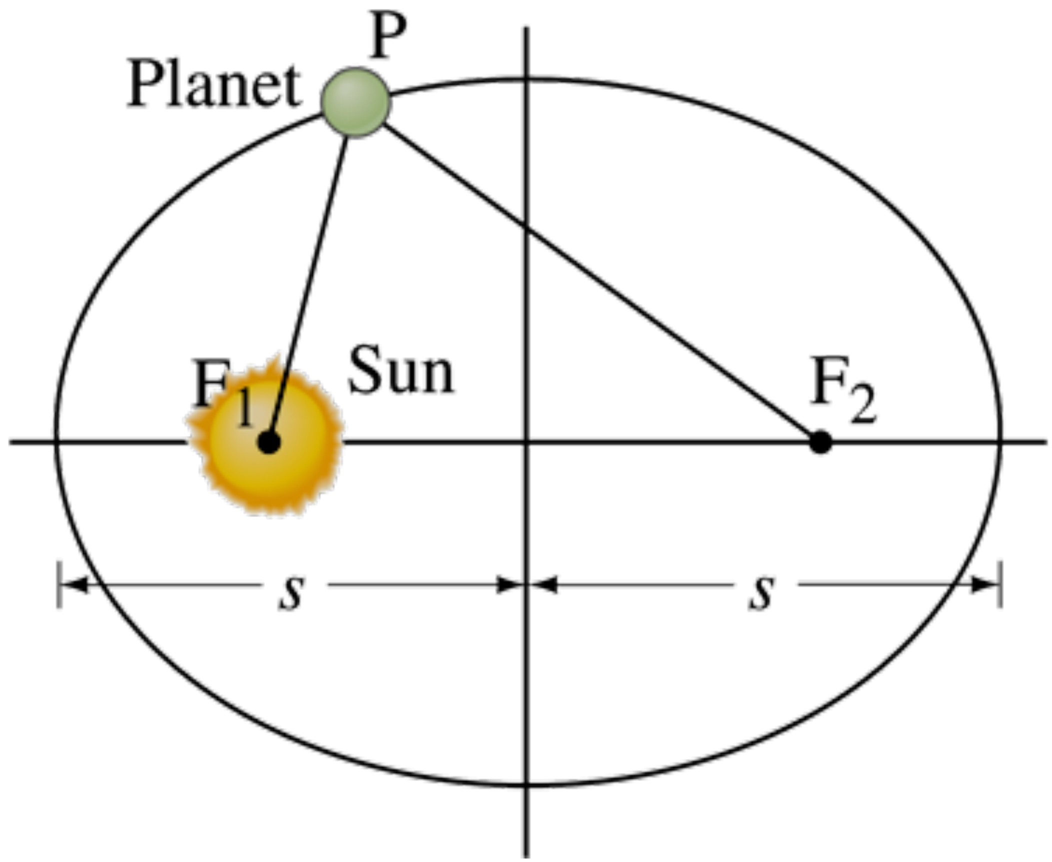
3 empirical findings we now refer to as Kepler's laws of planetary motion

Kepler’s laws provided basis for Newton's discovery of law of gravitation

# Kepler's law and Newton's Synthesis (cont'd)

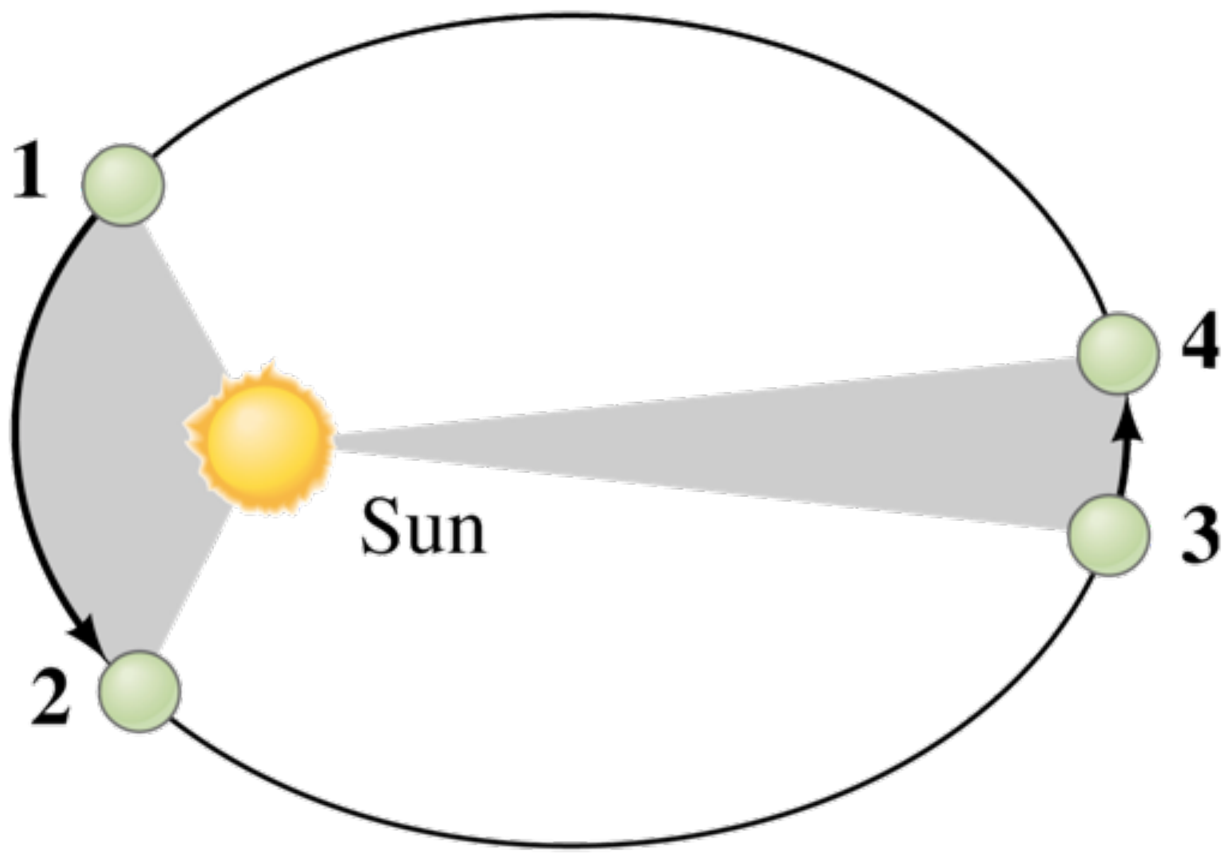
Kepler's laws describe planetary motion

1. Orbit of each planet is an ellipse with Sun at one focus



# Kepler's law and Newton's Synthesis (cont'd)

2. An imaginary line drawn from each planet to Sun sweeps out equal areas in equal times



# Kepler's law and Newton's Synthesis (cont'd)

Square of planet's orbital period

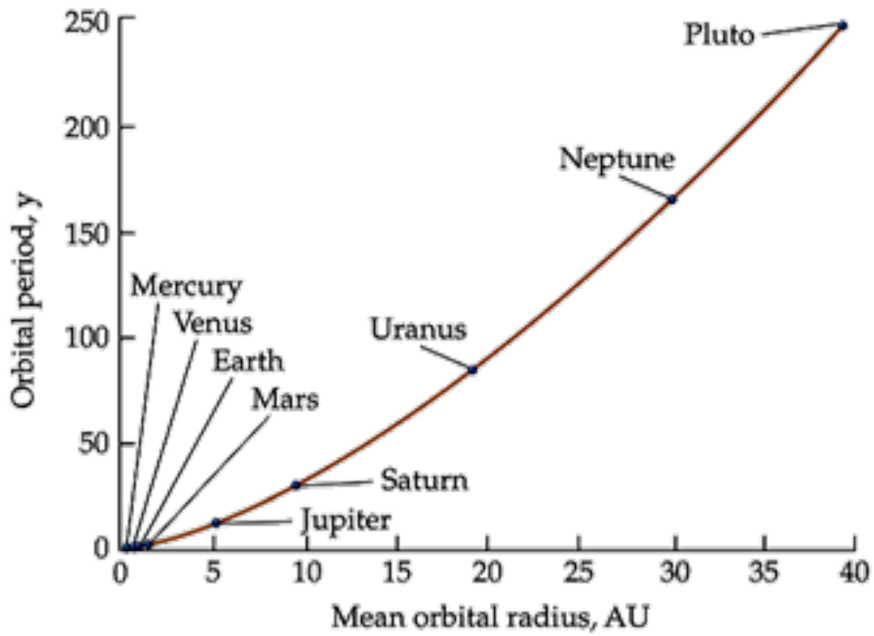
is proportional to cube of its mean distance from Sun

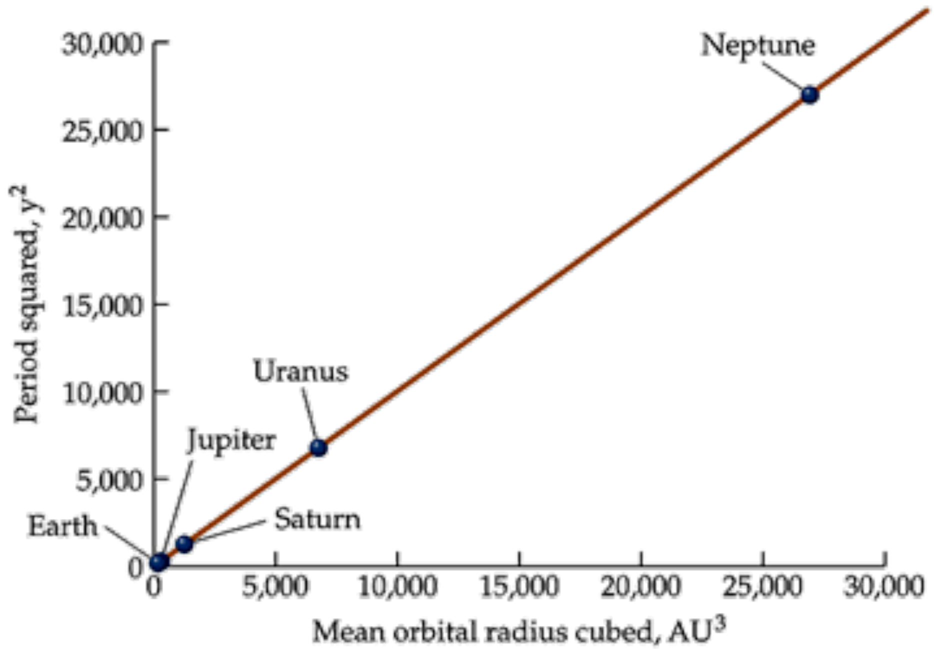
## Planetary data applied to Kepler's Third Law

| Planet | Mean Distance from Sun, $s$ ($10^6$ km) | Period, $T$ (Earth years) | $s^3/T^2$ ($10^{24}$ km$^3$/y$^2$) |
|---|---|---|---|
| Mercury | 57.9 | 0.241 | 3.34 |
| Venus | 108.2 | 0.615 | 3.35 |
| Earth | 149.6 | 1.0 | 3.35 |
| Mars | 227.9 | 1.88 | 3.35 |
| Jupiter | 778.3 | 11.86 | 3.35 |
| Saturn | 1427 | 29.5 | 3.34 |
| Uranus | 2870 | 84.0 | 3.35 |
| Neptune | 4497 | 165 | 3.34 |
| Pluto | 5900 | 248 | 3.34 |

# Kepler's law and Newton's Synthesis (cont'd)

Periods of planets as a function of mean orbital radii




Square of periods of planets versus cubes of mean orbital radii

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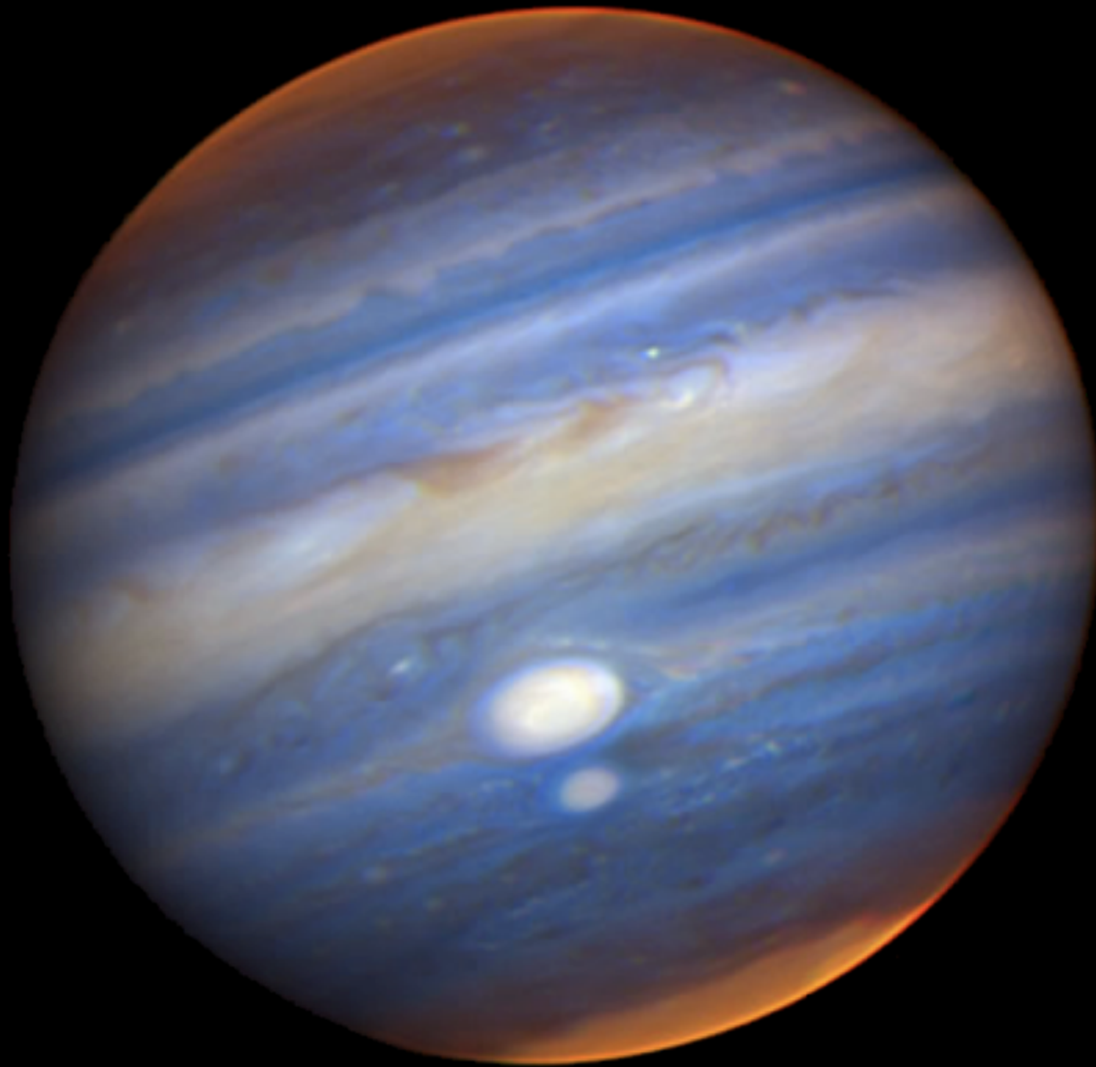
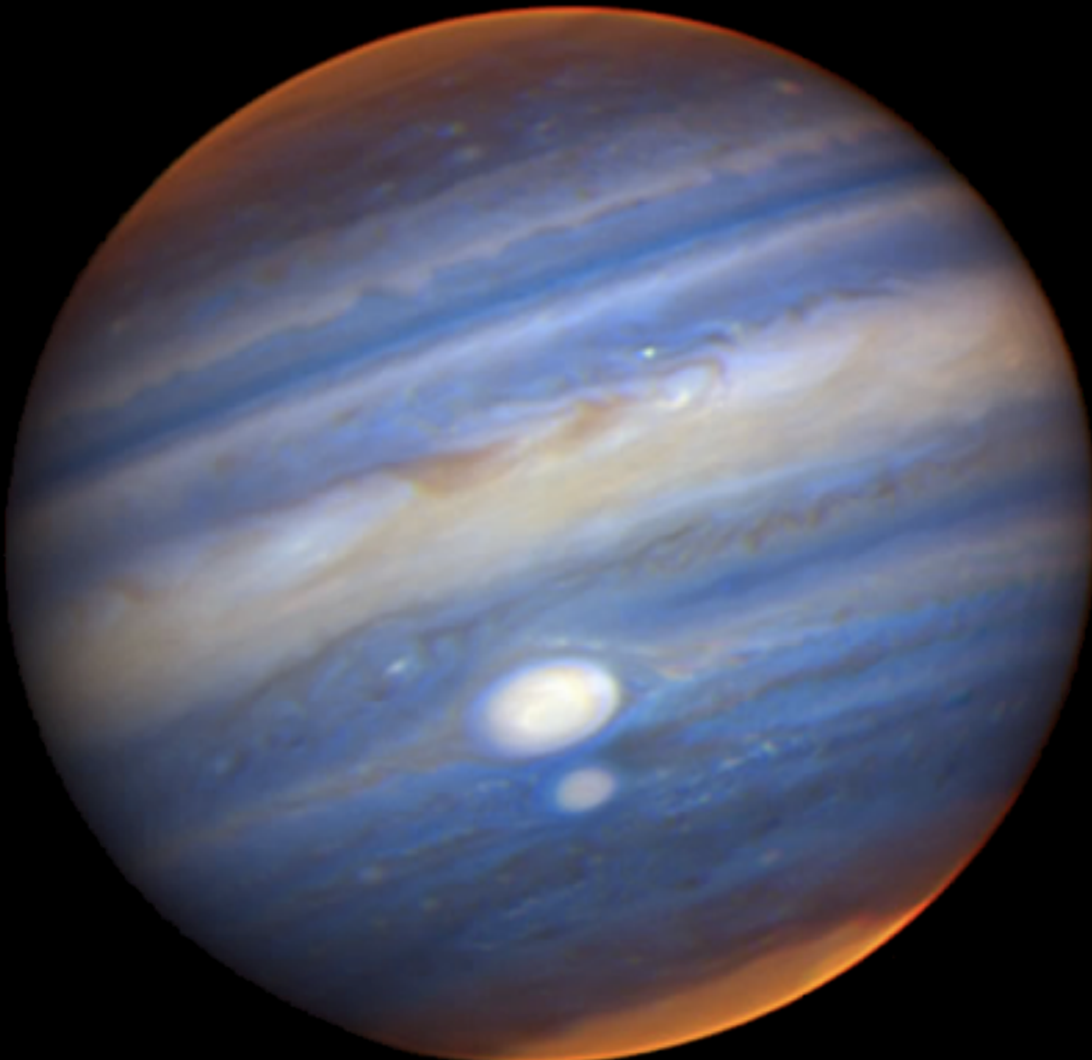
# Jupiter's orbit

Jupiter's mean orbital radius is 5.2 AU.
What is period of Jupiter's orbit around Sun?



# Jupiter's orbit

**Use Kepler's 3rd law to relate Jupiter's period to its mean orbital radius**



$$T_{\rm J}^2 = c r_{\rm SJ}^3$$

**using well-known Earth relation**

$$T_\oplus^2 = c r_{\rm SE}^3$$

$$\frac{T_{\rm J}^2}{T_\oplus^2} = \frac{r_{\rm SJ}^3}{r_{\rm SE}^3} \Rightarrow T_{\rm J} = T_\oplus \left(\frac{r_{\rm SJ}}{r_{\rm SE}}\right)^{3/2} = 11.9 \text{ yr}$$

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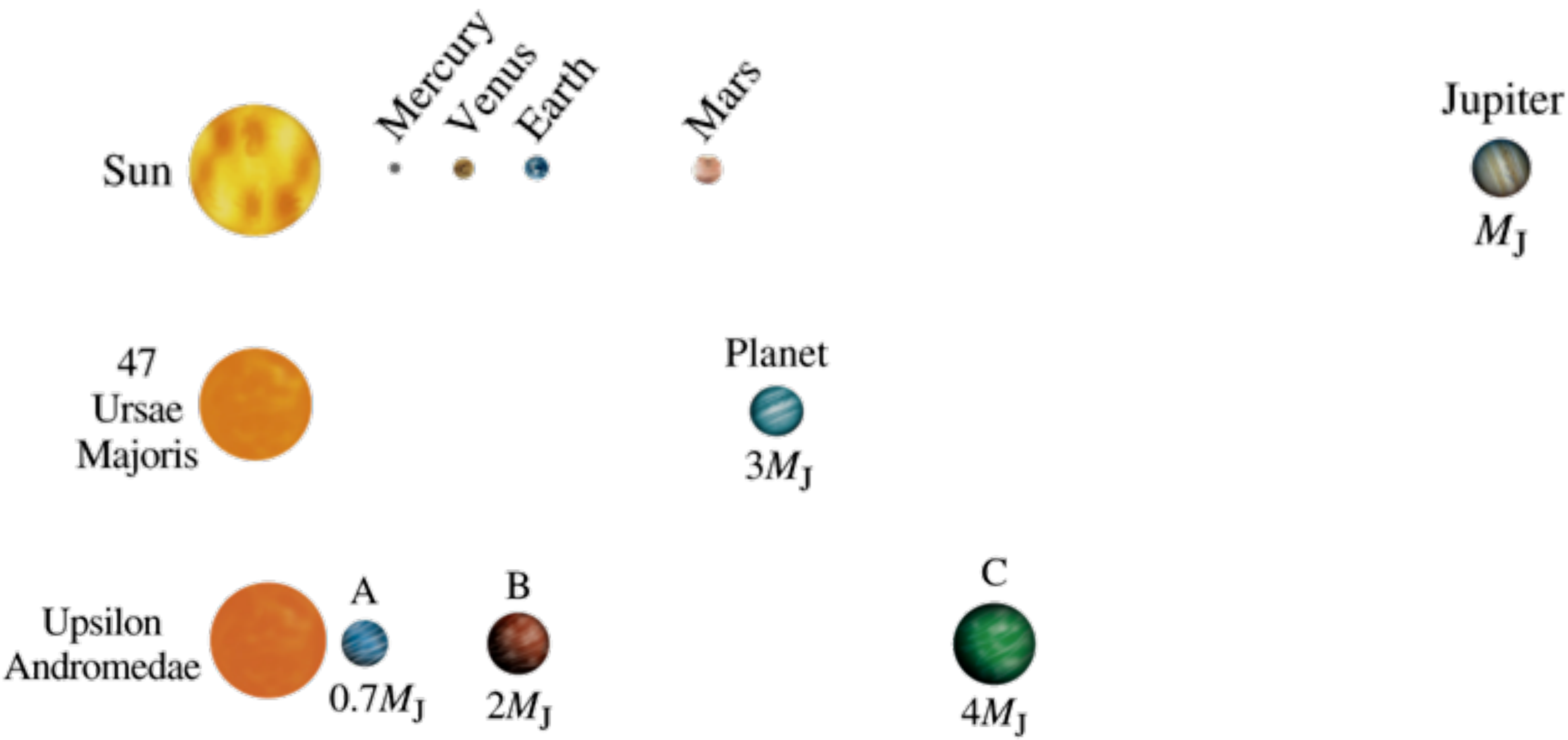
# Kepler's Laws from Newton's Law

Kepler's laws can be derived from Newton's laws

Irregularities in planetary motion led to discovery of Neptune

and irregularities in stellar motion
have led to discovery of many planets outside our Solar System

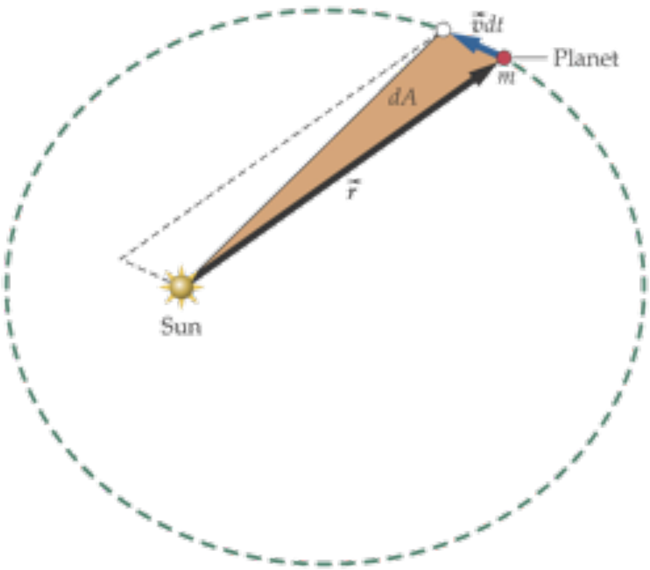


# Derivation of Kepler Second Law

In a time $dt$ planet moves a distance $\vec{v}\, dt$

and radius vector $\vec{r}$ sweeps out area shaded in figure

This is half area of parallelogram form by vectors $\vec{r}$ and $\vec{v}\, dt$




$$dA = \frac{1}{2} |\vec{r} \times \vec{v}\, dt| = \frac{|\vec{r} \times m\vec{v}|}{2\, m}\, dt$$

$$\Downarrow$$

$$\frac{dA}{dt} = \frac{L}{2\, m}$$

$L$ is angular momentum of planet around Sun

Because force of planet is along line from planet to Sun

it exerts no torque about Sun

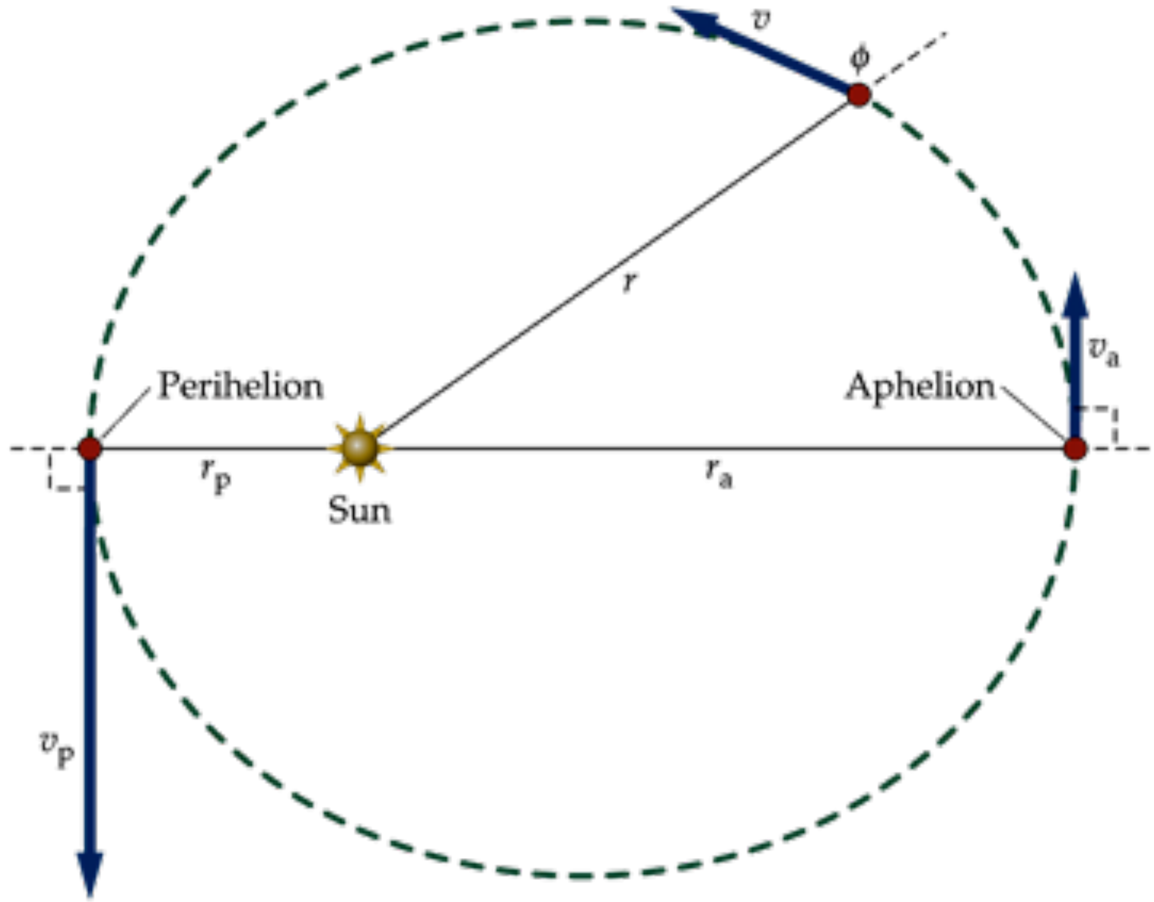
$$\Downarrow$$

$L$ is constant

Rate at which area is swept out is same for all parts of orbit

# Derivation of Kepler Second Law (cont'd)

$$L = \text{constant} \Rightarrow rv \sin\phi = \text{constant}$$




At aphelion and perihelion $\phi = 90^\circ \Rightarrow r_a\, v_a = r_p\, v_p$

# Newton's Law of Gravitation implies Kepler's Third Law for circular orbit

Planet Earth moves with speed $v$ in approximate circular orbit of radius $r$ about Sun

Gravitational force on Earth by Sun provides centripetal acceleration $v^2/r$

$$\frac{G M_\oplus \, M_\odot}{r^2} = M_\oplus \frac{v^2}{r}$$

$$v^2 = \frac{G M_\odot}{r}$$

Because Earth moves a distance $2\pi \, r$ in time $\tau$ its speed is related to period by

$$v = \frac{2\pi \, r}{\tau}$$

Substituting

$$\tau^2 = \frac{4\pi^2}{G M_\odot} r^3$$

Through the brilliance of Saturn's rings

Cassini caught a glimpse of a far-away planet and its moon

At a distance of just under 900 million miles

Earth shines bright among the many stars in the sky

distinguished  by its bluish tint


Gravitational
field
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# Gravitational Potential Energy

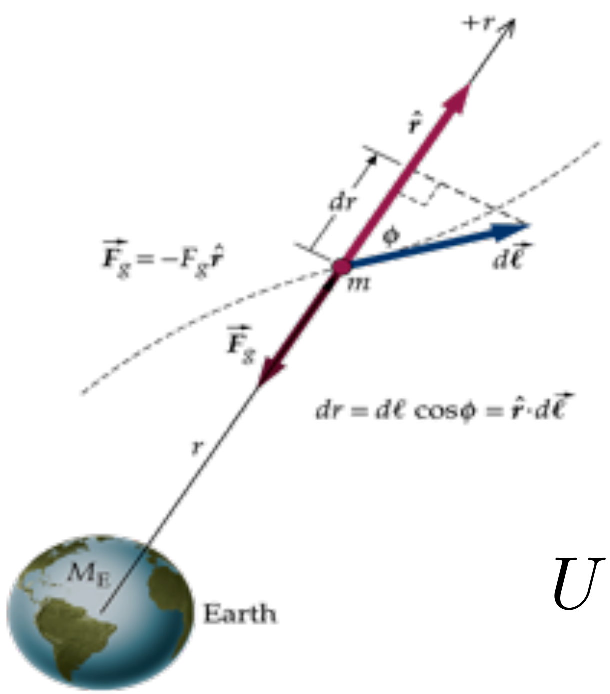
General definition of potential energy is

$$dU = -\vec{F} \cdot d\vec{l}$$

Where $\vec{F}$ is a conservative force acting on a particle

and $d\vec{l}$ is a general displacement of particle

$$dU = -\vec{F} \cdot d\vec{l} = -(F_g\,\hat{r}) \cdot d\vec{l} = \frac{GM_{\oplus}m}{r^2}dr$$

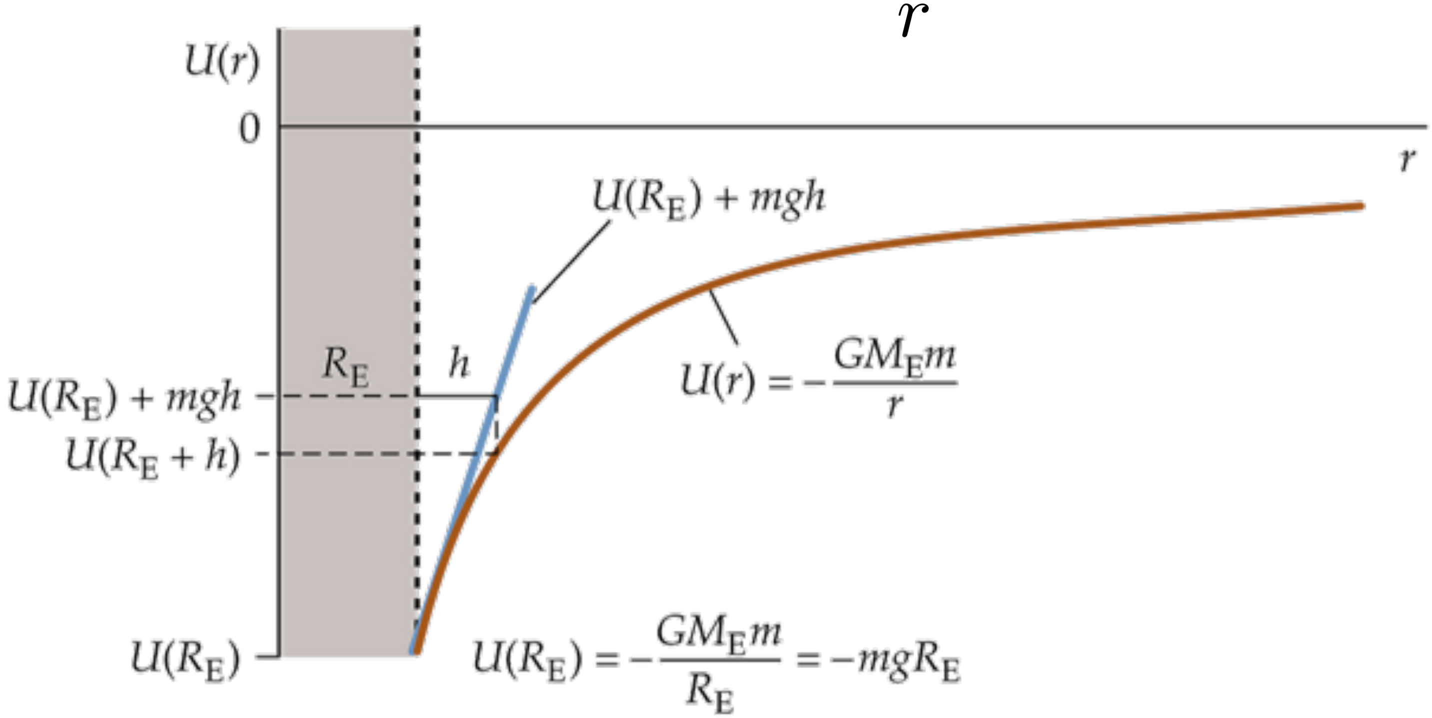



$$U = GM_{\oplus}m \int r^{-2}\,dr = -\frac{GM_{\oplus}m}{r} + U_0$$

# Gravitational Potential Energy (cont'd)

We can set $U_0 = 0$ and then $U = 0$ at $r \to \infty$

$$U = -\frac{GM_\oplus m}{r}$$




Gravitational potential of Earth starting at surface of planet

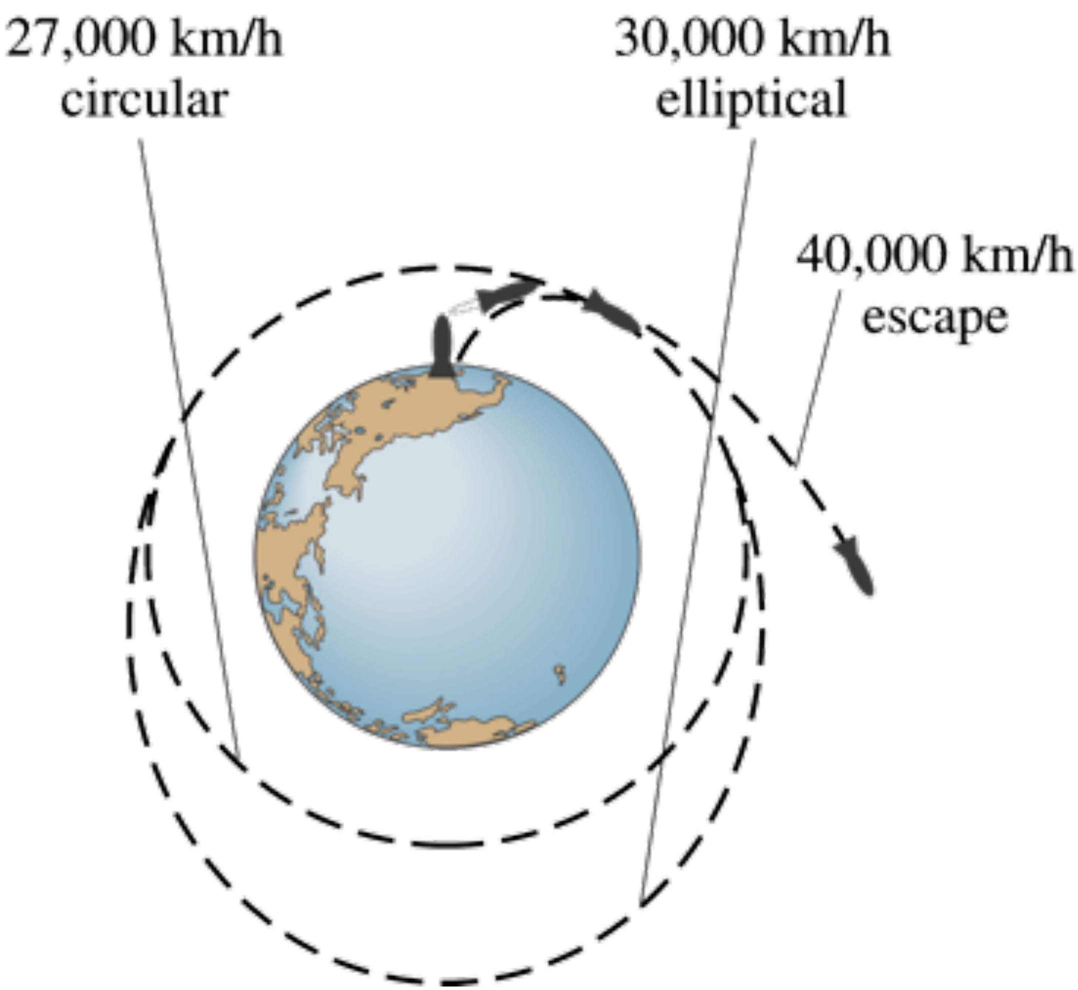
$$U = -\frac{GM_\oplus m}{R_\oplus}$$

and increasing with rising $r$

# Satellites

Satellites are routinely put into orbit around Earth

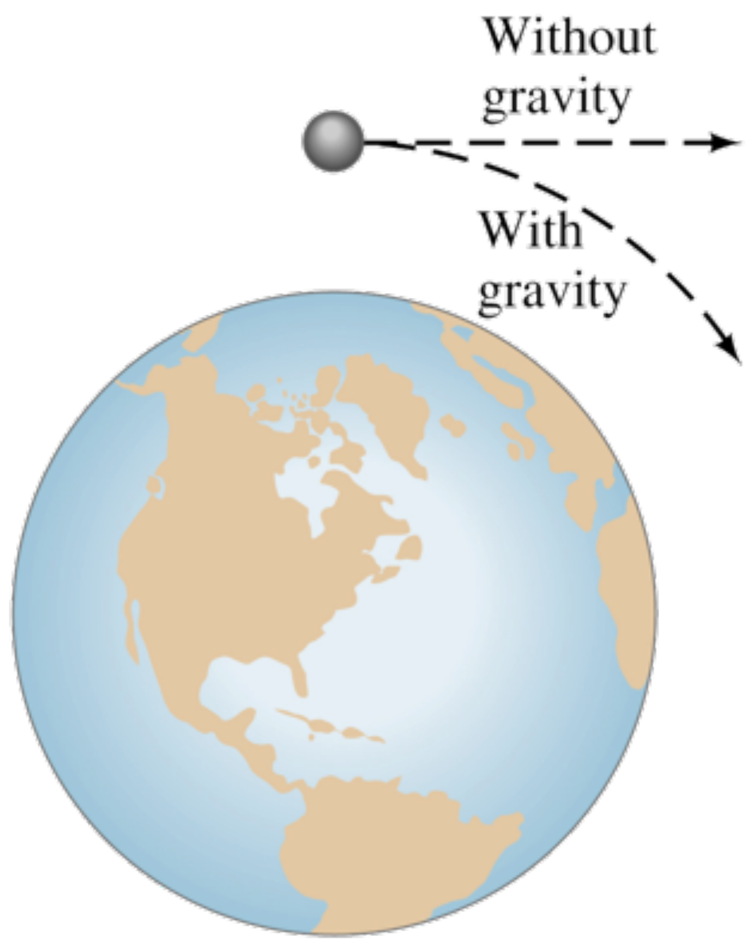
Tangential speed must be high enough for satellite not to return to Earth but not so high that it escapes Earth's gravity altogether



# Satellites (cont'd)

Satellite is kept in orbit by its speed

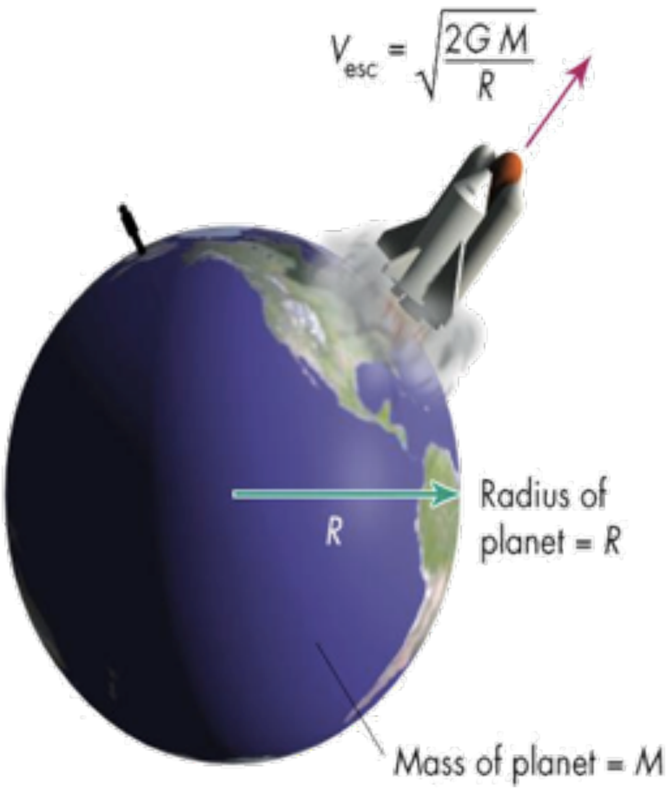
it is continually falling but Earth curves from underneath it



# Escape speed

$$K_f + U_f = K_i + U_i$$

Speed near Earth's surface corresponding to zero total energy is called escape speed $v_e$




$$0 = \frac{1}{2} m v_e^2 - \frac{G M_\oplus m}{R_\oplus}$$

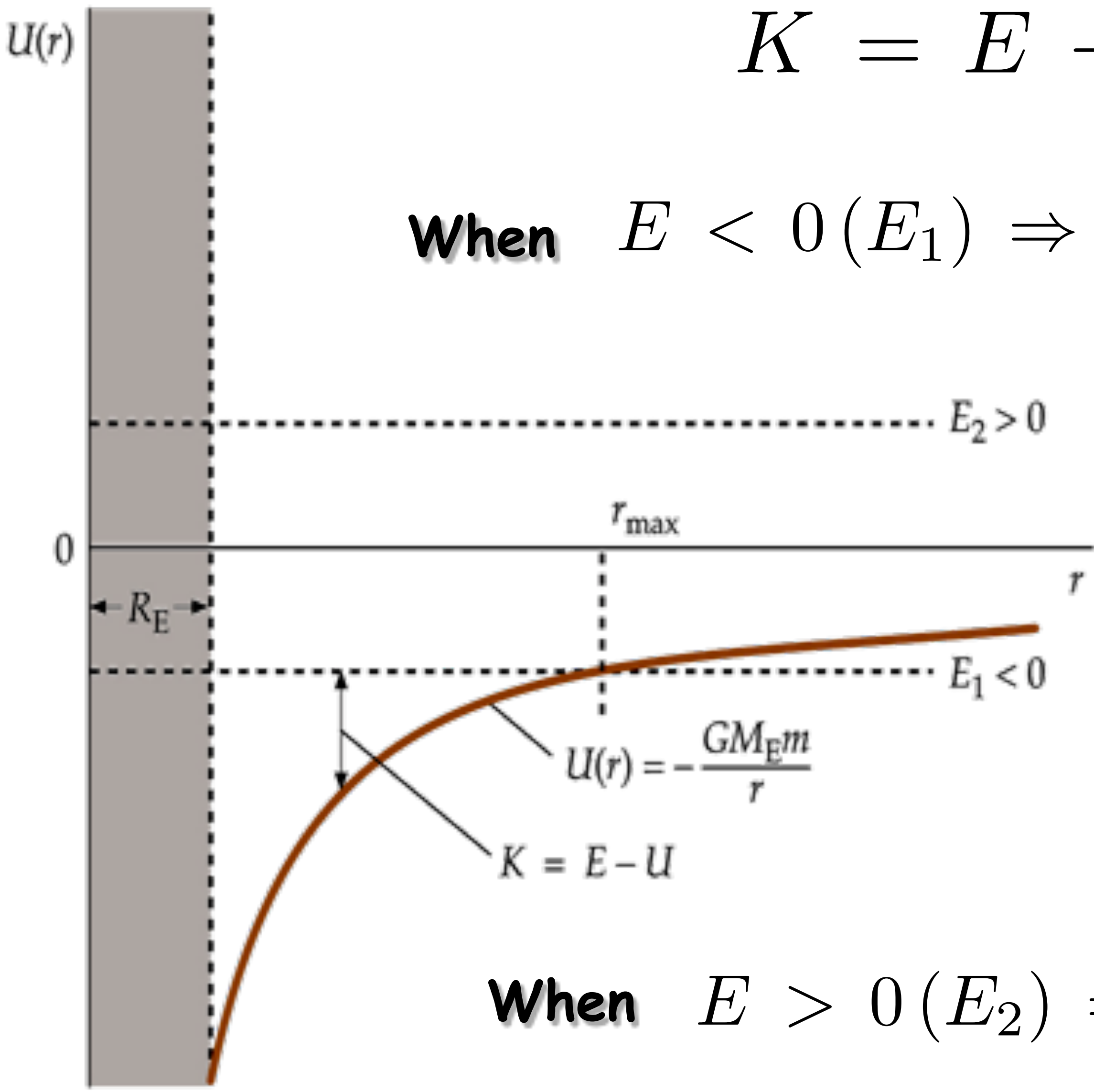
$$v_e = \left(\frac{2GM_\oplus}{R_\oplus}\right)^{\frac{1}{2}} = 11.2\, km/s$$

# Escape speed (cont'd)

Kinetic energy of an object at a distance r from center of Earth is

$$K = E - U(r)$$

When $E < 0\,(E_1) \Rightarrow K = 0$ at $r = r_{\max}$




When $E > 0\,(E_2) \Rightarrow$ object can escape Earth

# Gravitational field

Gravitational field at point $P$ is determined by placing a point particle of mass $m$ and calculating force on it due to all other particle

$$\vec{g} = \lim_{m \to 0} \frac{\vec{F}_g}{m}$$

Gravitational field at a point due to masses of a collection of point particles is vector sum of fields due to individual masses

$$\vec{g} = \sum_i \vec{g}_i$$

Locations of these point are called source point

To determine gravitational field of a continuous object find field $d\vec{g}$ due to small element of volume with mass dm and integrate over entire mass distribution of object

$$\vec{g} = \int d\vec{g}$$

Gravitational field of Earth at a distance $r \geq R_\oplus$ points towards Earth and has a magnitude ☛ $g(r) = \frac{GM_\oplus}{r^2}$

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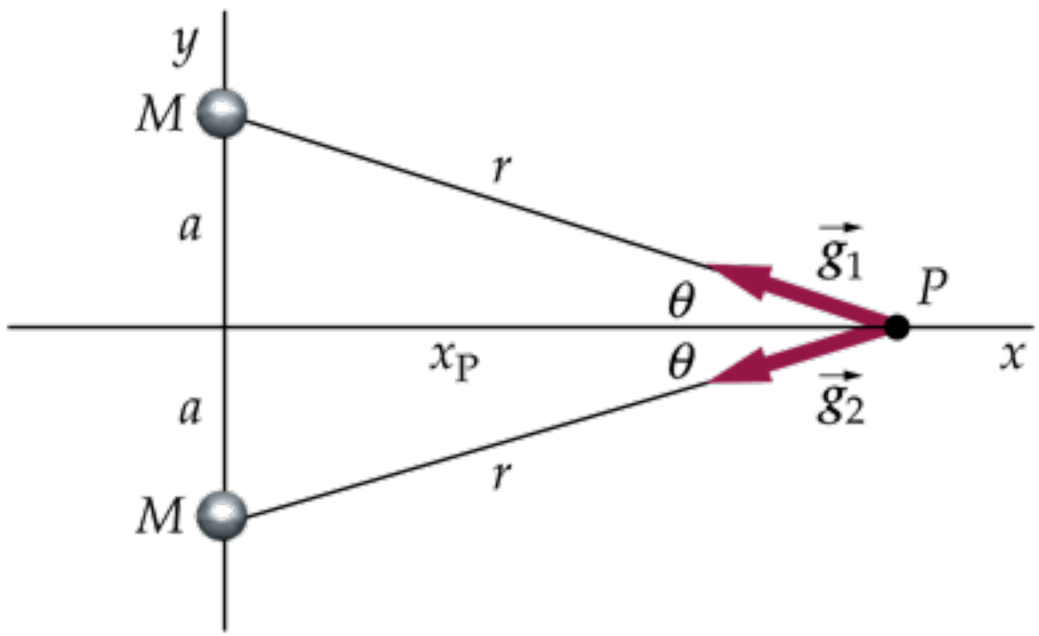
# Gravitational field of two point particles

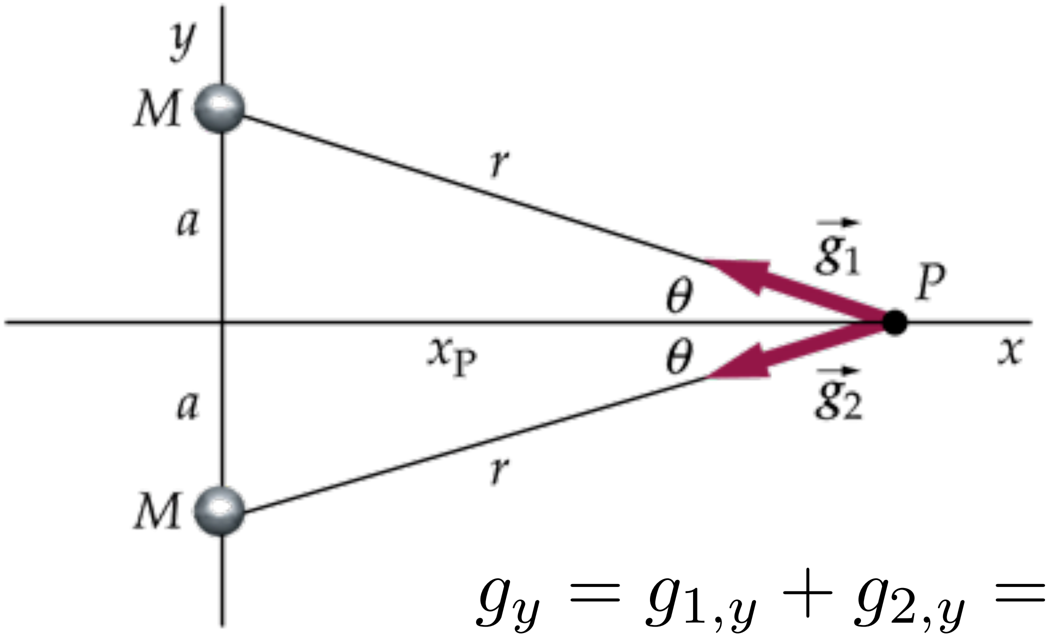
Two point particles each of mass $M$

are fixed in position on $y$ axis at $y = +a$ and $y = -a$

Find gravitational field at all points in $x$ axis as a function of $x$



# Gravitational field of two point particles




$$\vec{g}_1 = \vec{g}_2 = \frac{GM}{r^2}$$

$$g_y = g_{1,y} + g_{2,y} = g_1 \sin\theta - g_2 \sin\theta = 0$$

$$g_x = g_{1,x} + g_{2,x} = g_1 \cos\theta + g_2 \cos\theta = 2 g_1 \cos\theta$$

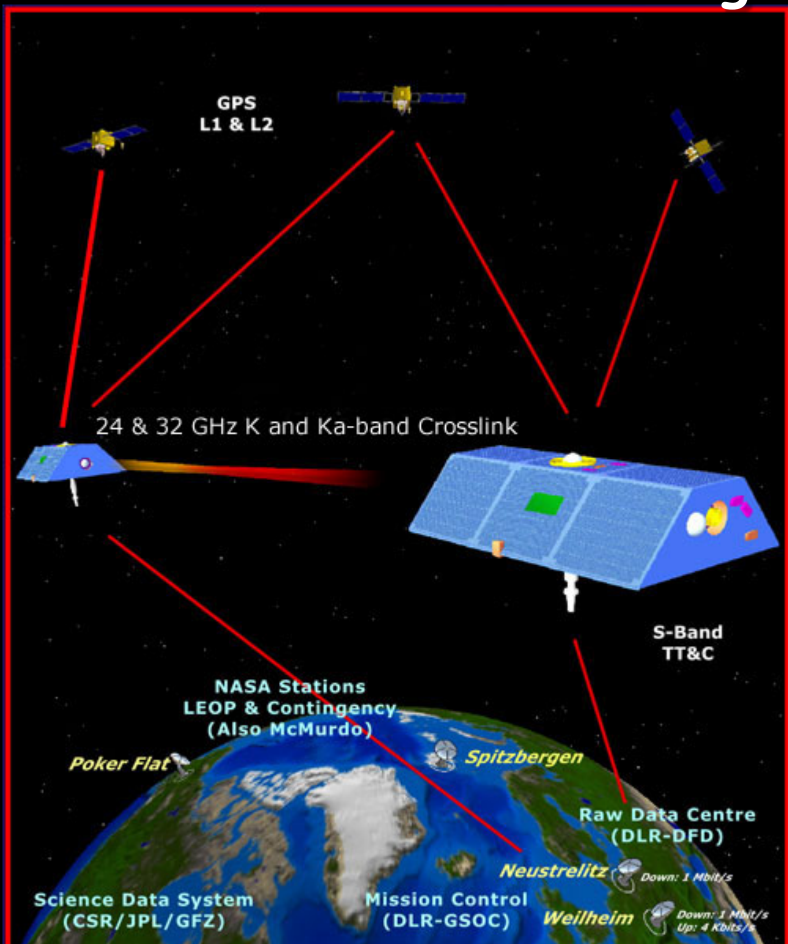
$$\cos\theta = x_P/r \Rightarrow \vec{g}(x_P) = -\frac{2GMx_P}{r^3}\hat{\imath} = -\frac{2GMx_P}{(x_P^2 + a^2)^{3/2}}\hat{\imath}$$

**For arbitrary x ☛** $\vec{g} = -\dfrac{2GMx}{(x^2 + a^2)^{3/2}}\,\hat{\imath}$

# A gravity map of Earth

Gravity Recovery and Climate Experiment (GRACE) is first mission in NASA's Earth System Science Pathfinder project which uses satellite-borne instrumentation to aid research on global climate change

Twin satellites launched in March 2002 are making detailed measurements of Earth gravitational field




They are in identical orbits with one satellite directly in front of other by about 220 km

Distance between satellites is continuously monitored with micrometer accuracy using onboard microwave telemetry equipment

How does distance between two satellites changes as satellites approach a region of increased mass?

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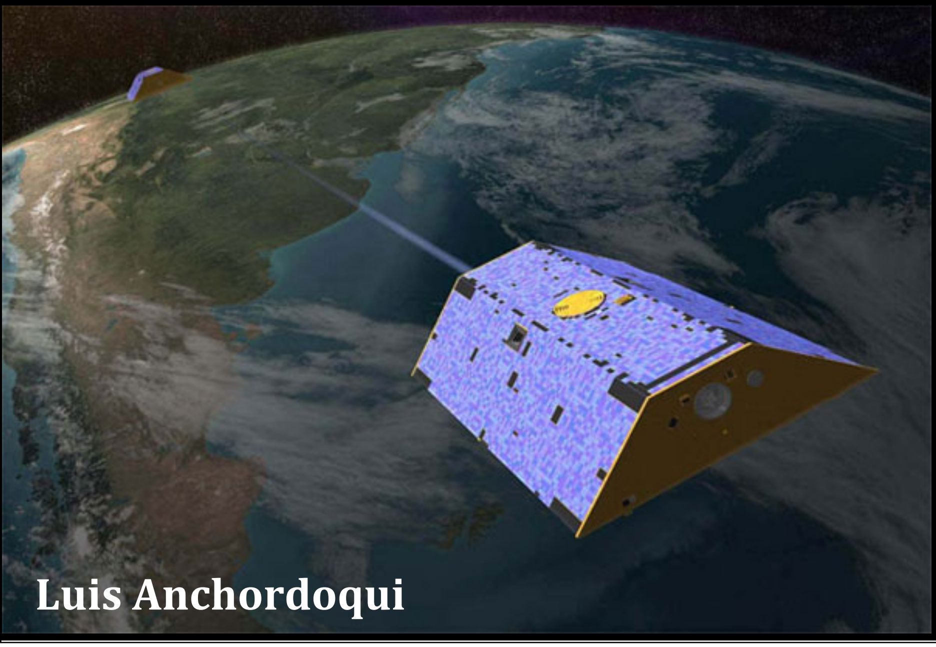
# A gravity map of Earth (Cont'd)

As twin satellites approach a region where there is an excess of mass increased gravitational field strength due to excess mass pulls them forward (toward excess of mass)

Pull on leading satellite is greater than pull on trailing satellite because leading satellite is closer to excess mass

Consequently leading satellite is gaining speed more rapidly than is trailing satellite



This results in an increase in separation between satellites

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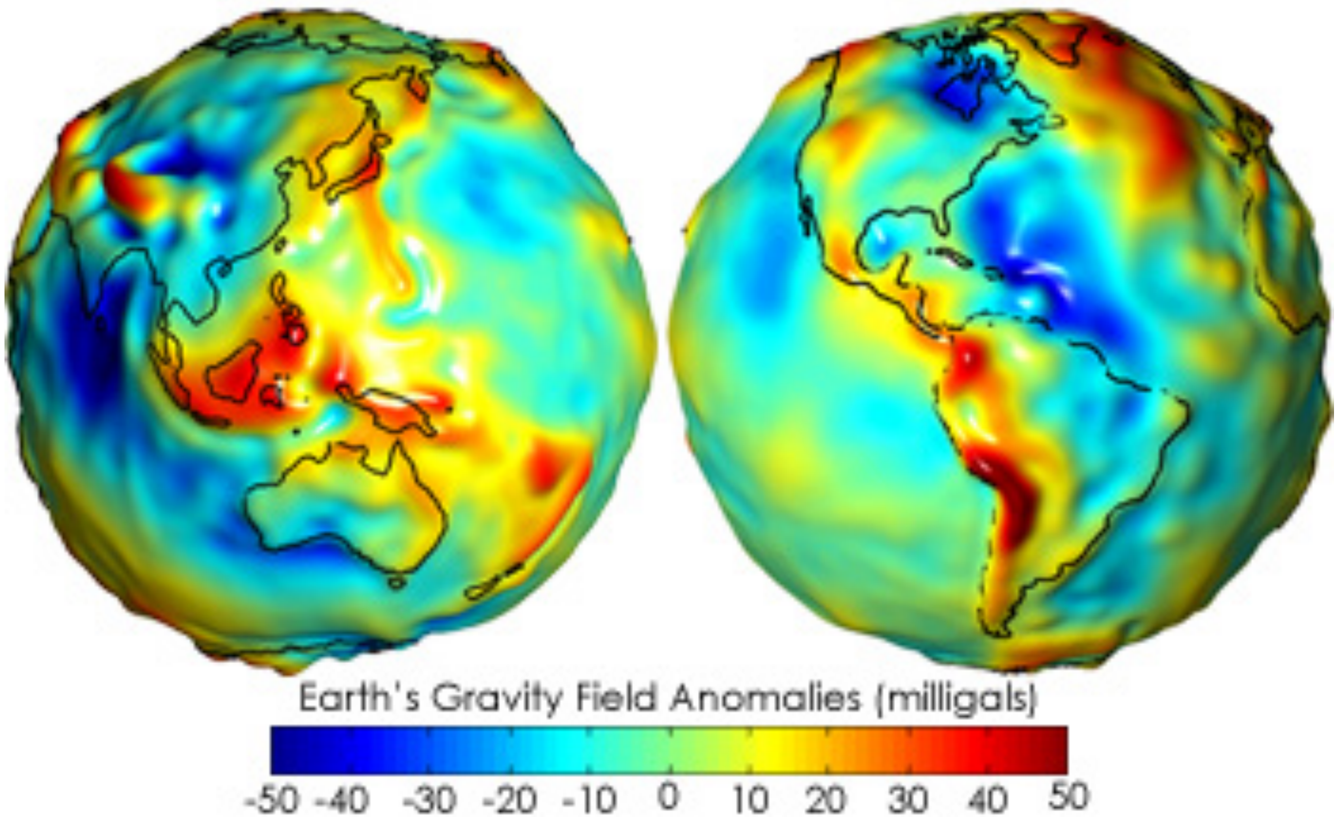
# A gravity map of Earth (Cont'd)

Gravity anomaly maps show how much Earth's actual gravity field differs from gravity field of a uniform, featureless Earth surface

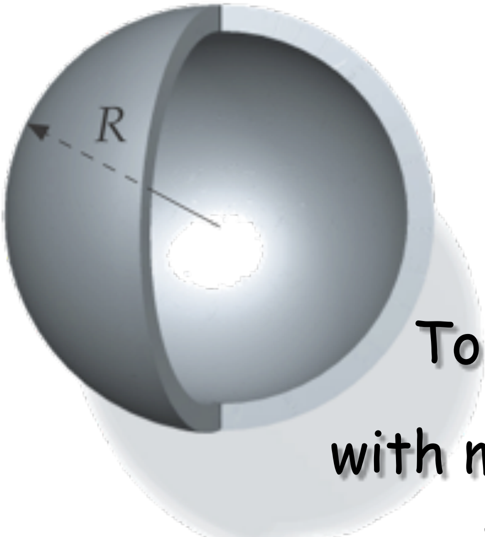
Anomalies highlight variations in strength of gravitational force over surface of Earth




Gal (sometimes called galileo) is defined as ☛ $1\,gal = 1\,cm/s^2$

# $\vec{g}$ of a spherical shell of a solid sphere

Consider a uniform spherical shell of mass $M$ and radius $R$



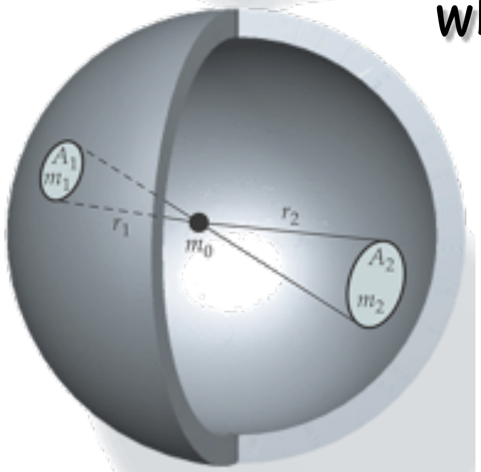

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \qquad r > R$$

$$\vec{g} = 0 \qquad r < R$$

To understand this last result consider shell segments

with masses $m_1$ & $m_2$ that are proportional to areas $A_1$ & $A_2$

which in turn are proportional to radii $r_1$ & $r_2$



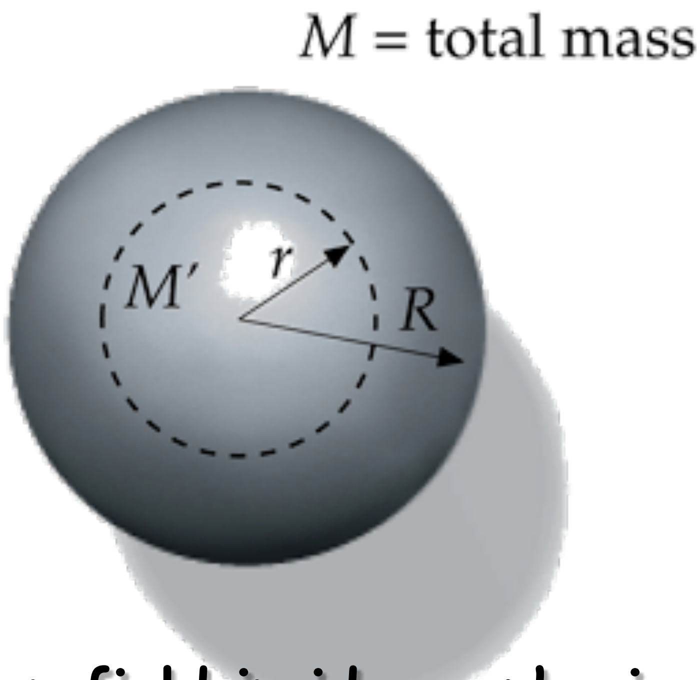

$$\frac{m_1}{m_2} = \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}$$

$$\frac{m_1}{r_1^2} = \frac{m_2}{r_2^2}$$

Because gravitational force falls off inversely as square of distance force on $m_0$ due to smaller mass $m_1$ on left, is exactly balanced by that due to more distant but larger mass $m_2$ on right

# $\vec{g}$ inside a solid sphere

Consider a uniform solid sphere of radius R and mass M




As we have seen field inside a spherical shell is zero

⇩

Mass of sphere outside r exerts no force inside r

Only mass M' whithin radius r contributes to gravitational field at r

# $\vec{g}$ inside a solid sphere (cont'd)

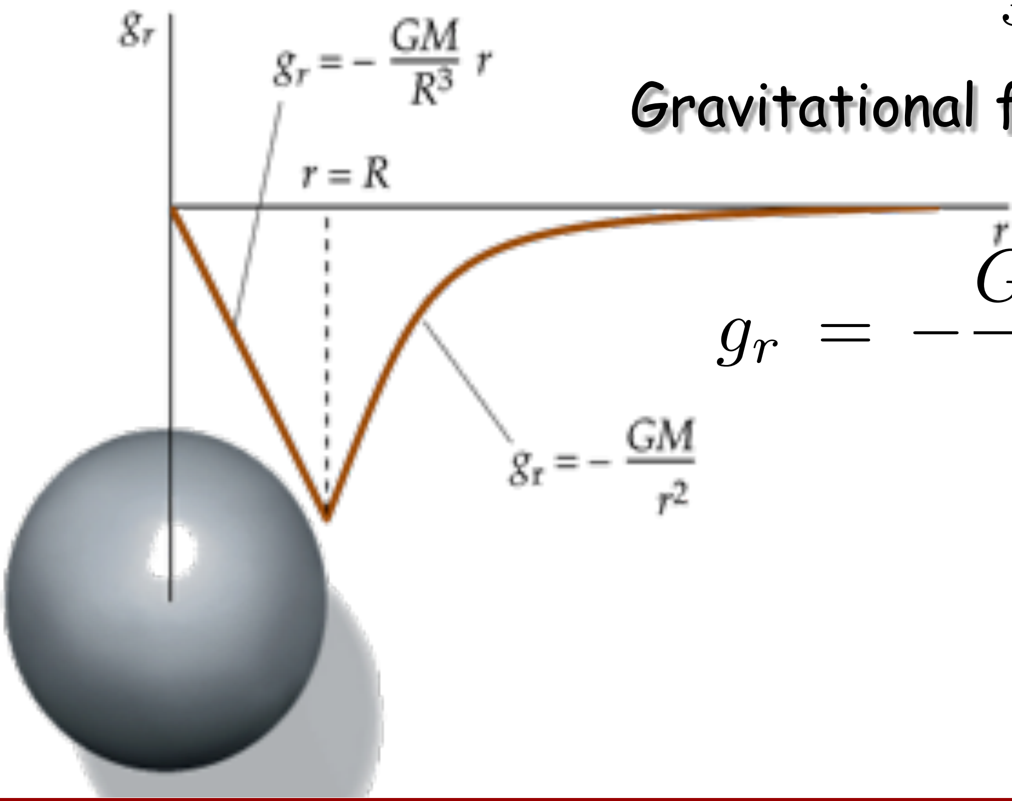
Mass inside r produces a field

equal to that of a point mass M' at center of sphere

For a uniform sphere

$$M' = \frac{\frac{4}{3}\pi\, r^3}{\frac{4}{3}\pi\, R^3}\, M$$

Gravitational field at a distance $r < R$ is then




$$g_r = -\frac{GM'}{r^2} = -\frac{GM}{r^2}\,\frac{r^3}{R^3} = -\frac{GMr}{R^3}$$

# Radially Dependent Density

A solid sphere of radius $R$ and mass $M$ is spherically symmetric but not uniform

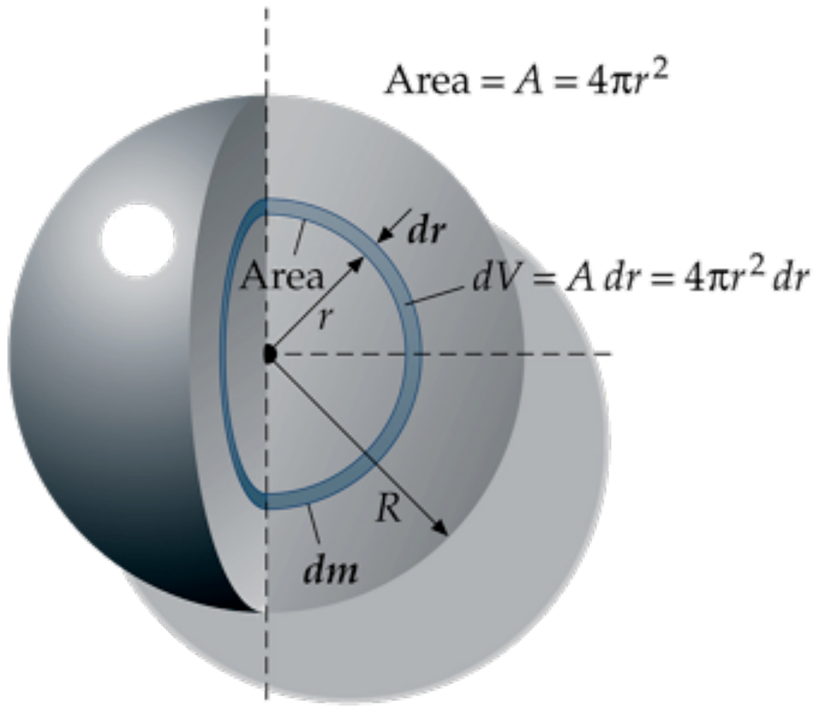
Its density $\rho$ is proportional to distance $r$ from center for $r < R$

That is, $\rho = Cr$ for $r < R$ where $C$ is a constant

ⓐ Find $C$

ⓑ Find $\vec{g}$ for all $r < R$

ⓒ Find $\vec{g}$ at $r = R/2$

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# Radially Dependent Density

(a)

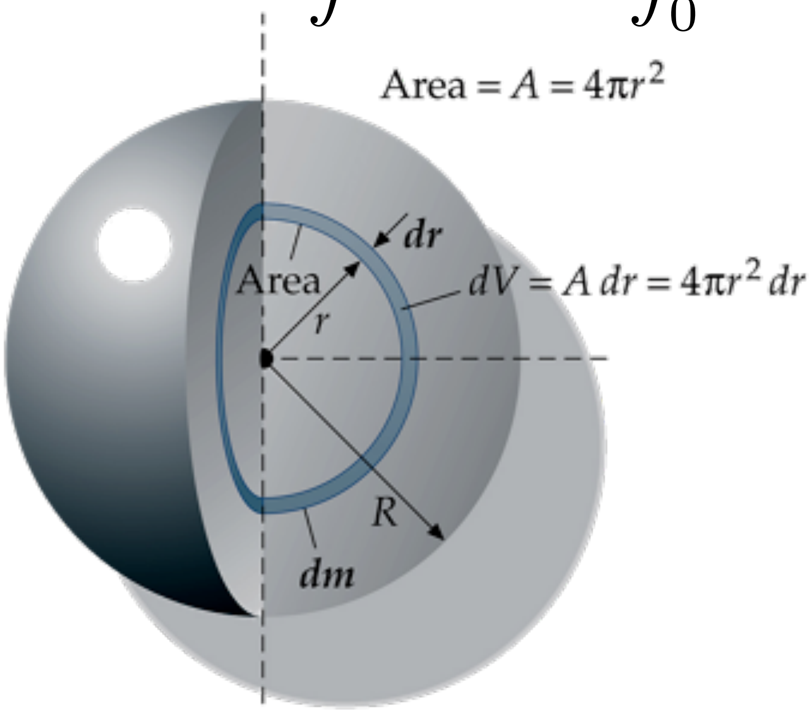
$$M = \int dM = \int \rho dV = \int_0^R Cr(4\pi r^2 dr) = C\pi R^4 \Rightarrow C = \frac{M}{\pi R^4}$$

(b)

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \; (r > R)$$

(c)

$$M' = \int \rho dV = \int_0^{R/2} Cr(4\pi r^2 dr) = \frac{1}{16}C\pi R^4 \Rightarrow M' = \frac{M}{16}$$




$$\vec{g} = -\frac{GM'}{r^2}\hat{r}$$

$$\vec{g} = -\frac{GM}{4R^2}\hat{r} \; (@r = R/2)$$

# Ocean tides

Ocean tides have long been of interest to humans

Chinese explained tides as breathing of Earth

Galileo tried unsuccessfully to explain tides ☛ effect of Earth´s motion
* Around sun once a year
* On its own axis once a day

(could not account for timing of approximately two high tides each day)

Mariners have known for at least 4000 yr that tides are related to Moon's phases

Exact relationship ☛ hidden behind many complicated factors

**Newton finally gave an adequate explanation**

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# Ocean tides (Cont'd)

To moon

Ocean tides are caused by gravitational attraction of ocean

To sun

Calculation is complicated

↓

Surface of Earth is not an inertial system!

Earth's rotation

Timing of tidal events is related to

Revolution of moon around Earth

If moon was stationary in space ☛ tidal cycle would be 24 hours long

However

Moon is in motion revolving around Earth

1 revolution takes about 27 days

↓

Adding about 50 minutes to tidal cycle

# Ocean tides (Cont'd)

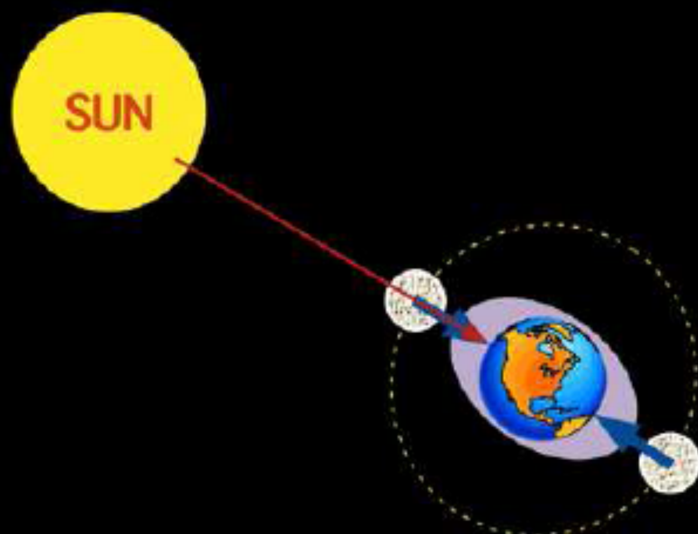
Second factor controlling tides on Earth's surface is then sun's gravity

Height of average solar tide is $\dfrac{M_\odot R^2_{\rm moon}}{M_{\rm moon} R^2_\odot} \approx 0.46$ average lunar tide

At certain times during moon's revolution around Earth

↓

Direction of its gravitational attraction is aligned with sun's




During these times two tide producing bodies act together

↓

Creating highest and lowest tides of year

Spring tides occur every 14-15 days during full and new moons

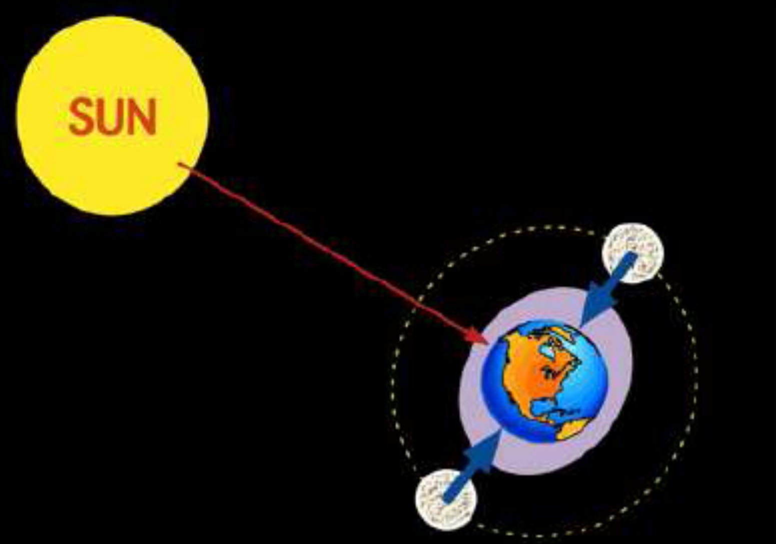
Luis Anchordoqui

# Ocean tides (Cont'd)

When gravitational pull of moon and sun are right angles to each other

↓

Daily tidal variations on Earth are at their least




Neap tides occur during first and last quarter of moon

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