Static Equilibrium

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Conditions for Equilibrium

An object with forces acting on it but that is not moving is said to be in equilibrium.
First condition for equilibrium is that forces along each coordinate axis add to zero.

\[ \sum F_y = F_A \sin 60^\circ - 200 \text{ kg} \times g = 0 \]

\[ F_A = 2260 \text{ N} \]

\[ \sum F_x = F_B - F_A \cos 60^\circ = 0 \]

\[ F_B = 1130 \text{ N} \]
Second condition of equilibrium is that there be no torque around any axis.

Choice of axis is arbitrary.
Center of Gravity

Point in a body around which the resultant torque due to gravity forces vanish

center of gravity average location of the weight of an object
Balancing a seesaw

A board of mass $M = 2$ kg serves as a seesaw for two children. Child A has a mass of 30 kg and sits 2.5 m from pivot point P (his CG is 2.5 m from the pivot).

At what distance $x$ from pivot must child B of mass 25 kg, place herself to balance the seesaw?

Assume board is uniform and centered over pivot.
Balancing a seesaw (cont’d)

\[ M\vec{g} \text{ and } F_N \text{ make no torque with respect to rotation point} \]

\[ \sum \tau = m_A \, g \, 2.5 \, m - m_B \, g \, x = 0 \Rightarrow x = 3 \, m \]
Consider right-hand (northernmost) section of Golden Gate bridge which has a length \( d_1 = 343 \text{ m} \)

Assume CG of this span halfway between tower and anchor

\( \text{a) Determine } F_{T1} \text{ and } F_{T2} \text{ (which act on the northernmost cable) in terms of } mg \text{ (weight of northernmost span)} \)

\( \text{b) calculate height } h \text{ needed for equilibrium} \)

Assume roadway is supported only by suspension cables, and neglect mass of cables and vertical wires
\[
\begin{align*}
\sum F_x &= F_{T_1} \cos 19^\circ - F_{T_2} \sin 60^\circ = 0 \\
\sum F_y &= F_{T_2} \cos 60^\circ - F_{T_1} \sin 19^\circ - mg = 0 \\
F_{T_2} &= F_{T_1} \frac{\cos 19^\circ}{\sin 60^\circ} \quad F_{T_1} = 4.5 \text{ mg} \\
F_{T_2} &= 5.0 \text{ mg} \\
\sum \tau &= mg \frac{1}{2} d_1 + F_{T_2,x} h - F_{T_2,y} d_1 = 0 \Rightarrow h = 158 \text{ m}
\end{align*}
\]
Kepler's law and Newton's Synthesis
Kepler's law and Newton's Synthesis

Nighttime sky with its myriad stars and shining planets has always fascinated people on Earth.

Towards end of XVI century astronomer Tycho Brahe studied motions of planets and made accurate observations.

Using Brahe's data Johannes Kepler worked out a detailed description of motion of planets about Sun.

3 empirical findings we now refer to as Kepler's laws of planetary motion.

Kepler’s laws provided basis for Newton’s discovery of law of gravitation.
Kepler's laws describe planetary motion

1. Orbit of each planet is an ellipse with Sun at one focus
2. An imaginary line drawn from each planet to Sun sweeps out equal areas in equal times.
Kepler's law and Newton's Synthesis (cont’d)

Square of planet's orbital period is proportional to cube of its mean distance from Sun

Planetary data applied to Kepler’s Third Law

<table>
<thead>
<tr>
<th>Planet</th>
<th>Mean Distance from Sun, $s$ $(10^6 \text{ km})$</th>
<th>Period, $T$ (Earth years)</th>
<th>$s^3/T^2$ $(10^{24} \text{ km}^3/\text{y}^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>57.9</td>
<td>0.241</td>
<td>3.34</td>
</tr>
<tr>
<td>Venus</td>
<td>108.2</td>
<td>0.615</td>
<td>3.35</td>
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<tr>
<td>Earth</td>
<td>149.6</td>
<td>1.0</td>
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<tr>
<td>Mars</td>
<td>227.9</td>
<td>1.88</td>
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<td>778.3</td>
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<tr>
<td>Pluto</td>
<td>5900</td>
<td>248</td>
<td>3.34</td>
</tr>
</tbody>
</table>
Kepler's law and Newton's Synthesis (cont’d)

Periods of planets as a function of mean orbital radii

Square of periods of planets versus cubes of mean orbital radii
Jupiter's mean orbital radius is 5.2 AU.
What is period of Jupiter's orbit around Sun?
Jupiter's orbit

Use Kepler's 3rd law to relate Jupiter's period to its mean orbital radius.

\[ T_J^2 = c r_{SJ}^3 \]

using well-known Earth relation

\[ T_\oplus^2 = c r_{SE}^3 \]

\[
\frac{T_J^2}{T_\oplus^2} = \frac{r_{SJ}^3}{r_{SE}^3} \implies T_J = T_\oplus \left( \frac{r_{SJ}}{r_{SE}} \right)^{3/2} = 11.9 \text{ yr}
\]

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Kepler’s Laws from Newton's Law

Kepler’s laws can be derived from Newton’s laws.

Irregularities in planetary motion led to discovery of Neptune and irregularities in stellar motion have led to discovery of many planets outside our Solar System.
Derivation of Kepler Second Law

In a time $dt$ planet moves a distance $\vec{v} dt$ and radius vector $\vec{r}$ sweeps out area shaded in figure.

This is half area of parallelogram form by vectors $\vec{r}$ and $\vec{v} dt$

\[ dA = \frac{1}{2} |\vec{r} \times \vec{v} dt| = \frac{|\vec{r} \times m \vec{v}|}{2m} dt \]

\[ \frac{dA}{dt} = \frac{L}{2m} \]

$L$ is angular momentum of planet around Sun.

Because force of planet is along line from planet to Sun, it exerts no torque about Sun.

$L$ is constant.

Rate at which area is swept out is same for all parts of orbit.
Derivation of Kepler Second Law (cont’d)

\[ L = \text{constant} \Rightarrow rv \sin \phi = \text{constant} \]

At aphelion and perihelion

\[ \phi = 90^\circ \Rightarrow r_a v_a = r_p v_p \]
Newton’s Law of Gravitation implies Kepler’s Third Law for circular orbit

Planet Earth moves with speed \( v \)
in approximate circular orbit of radius \( r \) about Sun

Gravitational force on Earth by Sun provides centripetal acceleration \( v^2/r \)

\[
\frac{GM\oplus M\odot}{r^2} = M\oplus \frac{v^2}{r}
\]

\[
v^2 = \frac{GM\odot}{r}
\]

Because Earth moves a distance \( 2\pi r \) in time \( \tau \) its speed is related to period by

\[
v = \frac{2\pi r}{\tau}
\]

Substituting

\[
\tau^2 = \frac{4\pi^2}{GM\odot} r^3
\]
**Gravitational Potential Energy**

*General definition of potential energy is*

\[ dU = -\vec{F} \cdot d\vec{l} \]

Where \( \vec{F} \) is a conservative force acting on a particle

and \( d\vec{l} \) is a general displacement of particle

\[ dU = -\vec{F} \cdot d\vec{l} = -(F_g \hat{r}) \cdot d\vec{l} = \frac{GM_{\oplus}m}{r^2}dr \]

\[ U = GM_{\oplus}m \int r^{-2} dr = -\frac{GM_{\oplus}m}{r} + U_0 \]
Gravitational Potential Energy (cont’d)

We can set $U_0 = 0$ and then $U = 0$ at $r \to \infty$

$$U = -\frac{GM_\oplus m}{r}$$

Gravitational potential of Earth starting at surface of planet

$$U(r) = -\frac{G M_E m}{r}$$

Gravitational potential of Earth starting at surface of planet

$$U = -\frac{GM_\oplus m}{R_\oplus}$$

and increasing with rising $r$
**Satellites**

Satellites are routinely put into orbit around Earth. Tangential speed must be high enough for the satellite not to return to Earth but not so high that it escapes Earth’s gravity altogether.

- **27,000 km/h** circular
- **30,000 km/h** elliptical
- **40,000 km/h** escape
Satellites (cont’d)

Satellite is kept in orbit by its speed

it is continually falling but Earth curves from underneath it
**Escape speed**

\[ K_f + U_f = K_i + U_i \]

Speed near Earth's surface corresponding to zero total energy is called **escape speed** \( v_e \)

\[
0 = \frac{1}{2} mv_e^2 - \frac{GM\oplus m}{R\oplus}
\]

\[
v_e = \left( \frac{2GM\oplus}{R\oplus} \right)^{\frac{1}{2}} = 11.2 \text{ km/s}
\]
Escape speed (cont’d)

Kinetic energy of an object at a distance $r$ from center of Earth is

$$ K = E - U(r) $$

When $E < 0 \ (E_1) \Rightarrow K = 0$ at $r = r_{\text{max}}$

When $E > 0 \ (E_2) \Rightarrow \text{object can escape Earth}$
Gravitational field

Gravitational field at point $P$ is determined by placing a point particle of mass $m$ and calculating force on it due to all other particle

$$\vec{g} = \lim_{m \to 0} \frac{\vec{F}_g}{m}$$

Gravitational field at a point due to masses of a collection of point particles is vector sum of fields due to individual masses

$$\vec{g} = \sum_i \vec{g}_i$$

Locations of these point are called source point

To determine gravitational field of a continuous object find field $d\vec{g}$ due to small element of volume with mass $dm$ and integrate over entire mass distribution of object

$$\vec{g} = \int d\vec{g}$$

Gravitational field of Earth at a distance $r \geq R_\oplus$ points towards Earth and has a magnitude $g(r) = \frac{GM_\oplus}{r^2}$
Gravitational field of two point particles

Two point particles each of mass \( M \) are fixed in position on the \( y \) axis at \( y = a \) and \( y = -a \). Find gravitational field at all points in the \( x \) axis as a function of \( x \).
Gravitational field of two point particles

\[ \vec{g}_1 = \vec{g}_2 = \frac{GM}{r^2} \]

\[ g_y = g_{1,y} + g_{2,y} = g_1 \sin \theta - g_2 \sin \theta = 0 \]

\[ g_x = g_{1,x} + g_{2,x} = g_1 \cos \theta + g_2 \cos \theta = 2g_1 \cos \theta \]

\[ \cos \theta = \frac{x_P}{r} \Rightarrow \vec{g}(x_P) = -\frac{2GMx_P}{r^3} \hat{i} = -\frac{2GMx_P}{(x_P^2 + a^2)^{3/2}} \hat{i} \]

For arbitrary \( x \) \( \vec{g} = -\frac{2GMx}{(x^2 + a^2)^{3/2}} \hat{i} \)
A gravity map of Earth

Gravity Recovery and Climate Experiment (GRACE) is first mission in NASA’s Earth System Science Pathfinder project which uses satellite-borne instrumentation to aid research on global climate change.

Twin satellites launched in March 2002 are making detailed measurements of Earth gravitational field.

They are in identical orbits with one satellite directly in front of the other by about 220 km.

Distance between satellites is continuously monitored with micrometer accuracy using onboard microwave telemetry equipment.

How does distance between two satellites changes as satellites approach a region of increased mass?
As twin satellites approach a region where there is an excess of mass, increased gravitational field strength due to excess mass pulls them forward (toward excess of mass).

Pull on leading satellite is greater than pull on trailing satellite because leading satellite is closer to excess mass.

Consequently leading satellite is gaining speed more rapidly than is trailing satellite.

This results in an increase in separation between satellites.
A gravity map of Earth (Cont’d)

Gravity anomaly maps show how much Earth’s actual gravity field differs from gravity field of a uniform, featureless Earth surface.

Anomalies highlight variations in strength of gravitational force over surface of Earth.

Gal (sometimes called galileo) is defined as

\[ 1 \text{ gal} = 1 \text{ cm/s}^2 \]
\textbf{\textit{\overrightarrow{g}} of a spherical shell of a solid sphere}

Consider a uniform spherical shell of mass $M$ and radius $R$

\[\overrightarrow{g} = -\frac{GM}{r^2} \hat{r} \quad r > R\]

\[\overrightarrow{g} = 0 \quad r < R\]

To understand this last result consider shell segments with masses $m_1$ & $m_2$ that are proportional to areas $A_1$ & $A_2$

which in turn are proportional to radii $r_1$ & $r_2$

\[
\frac{m_1}{m_2} = \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}
\]

Because gravitational force falls off inversely as square of distance force on $m_0$ due to smaller mass $m_1$ on left, is exactly balanced by that due to more distant but larger mass $m_2$ on right
Consider a uniform solid sphere of radius $R$ and mass $M$

As we have seen field inside a spherical shell is zero

Mass of sphere outside $r$ exerts no force inside $r$

Only mass $M'$ within radius $r$ contributes to gravitational field at $r$
Mass inside $r$ produces a field equal to that of a point mass $M'$ at center of sphere.

For a uniform sphere,

$$M' = \frac{4\pi}{3} \frac{r^3}{R^3} M$$

Gravitational field at a distance $r < R$ is then

$$g_r = -\frac{GM'}{r^2} = -\frac{GM}{r^2} \frac{r^3}{R^3} = -\frac{GMr}{R^3}$$
A solid sphere of radius $R$ and mass $M$ is spherically symmetric but not uniform. Its density $\rho$ is proportional to distance $r$ from center for $r < R$. That is, $\rho = Cr$ for $r < R$ where $C$ is a constant.

(a) Find $C$

(b) Find $\vec{g}$ for all $r < R$

(c) Find $\vec{g}$ at $r = R/2$
Radially Dependent Density

\[ M = \int dM = \int \rho dV = \int_0^R Cr(4\pi r^2 dr) = C\pi R^4 \Rightarrow C = \frac{M}{\pi R^4} \]

\[ \vec{g} = -\frac{GM}{r^2} \hat{r} \quad (r > R) \]

\[ M' = \int \rho dV = \int_0^{R/2} Cr(4\pi r^2 dr) = \frac{1}{16} C\pi R^4 \Rightarrow M' = \frac{M}{16} \]

\[ \vec{g} = -\frac{GM'}{r^2} \hat{r} \]

\[ \vec{g} = -\frac{GM}{4R^2} \hat{r} \quad (@r = R/2) \]
Ocean tides

Ocean tides have long been of interest to humans.

Chinese explained tides as breathing of Earth:
* Around sun once a year

Galileo tried unsuccessfully to explain tides by effect of Earth’s motion:
* On its own axis once a day (could not account for timing of approximately two high tides each day)

Mariners have known for at least 4000 yr that tides are related to Moon’s phases.

Exact relationship hidden behind many complicated factors.

Newton finally gave an adequate explanation.
Ocean tides (Cont’d)

Ocean tides are caused by gravitational attraction of ocean

To moon

Calculation is complicated

To sun

Surface of Earth is not an inertial system!

Earth’s rotation

Timing of tidal events is related to

Revolution of moon around Earth

If moon was stationary in space \[ \text{tidal cycle would be 24 hours long} \]

However

Moon is in motion revolving around Earth

1 revolution takes about 27 days

Adding about 50 minutes to tidal cycle
Ocean tides (Cont’d)
Second factor controlling tides on Earth’s surface is then sun’s gravity

Height of average solar tide is \( \frac{M_\odot R_{\text{moon}}^2}{M_{\text{moon}} R_\odot^2} \approx 0.46 \) average lunar tide

At certain times during moon’s revolution around Earth

Direction of its gravitational attraction is aligned with sun’s

During these times two tide producing bodies act together

Creating highest and lowest tides of year

Spring tides occur every 14-15 days during full and new moons
Ocean tides (Cont’d)

When gravitational pull of moon and sun are right angles to each other

↓

Daily tidal variations on Earth are at their least

Neap tides occur during first and last quarter of moon

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