# Lesson 7


Physics 168

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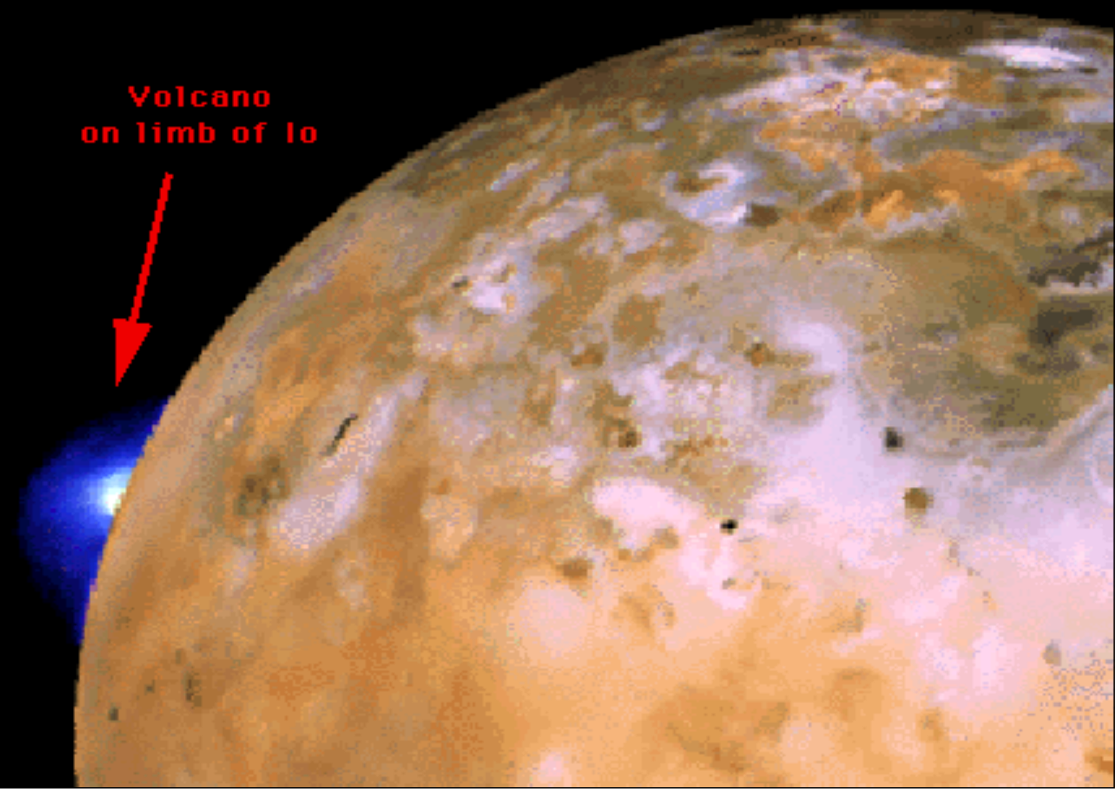
# Eruption of a large volcano on Jupiter’s moon

When volcano erupts speed of effluence exceeds escape speed of Io and so a stream of particles is projected into space

Material in stream can collide with and stick to surface of asteroid passing through stream

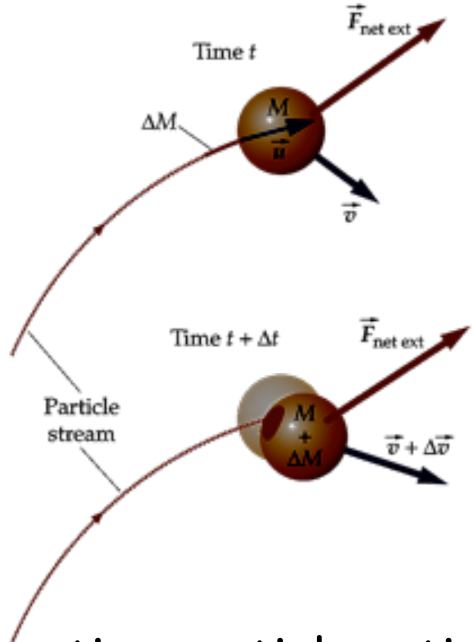
We now consider effect of impact of this material on motion of asteroid



# Continuously varying mass

Consider continuous stream of matter moving at velocity $\vec{u}$

which impacts object of mass $M$ that is moving with velocity $\vec{v}$




This impacting particles stick to object

increasing its mass by $\Delta M$ during time $\Delta t$

During time $\Delta t$ velocity $\vec{v}$ changes by $\Delta \vec{v}$

Applying impulse momentum theorem to this system

$$\vec{F}_{\text{net, ext}}\,\Delta t = \Delta\vec{P} = \vec{P}_{\text{f}} - \vec{P}_{\text{i}} = [(M + \Delta M)(\vec{v} + \Delta\vec{v})] - [M\vec{v} + \Delta M\,\vec{u}]$$

# Continuously varying mass (cont'd)

Rearranging terms

$$\vec{F}_{\rm net,\,ext}\,\Delta t \;=\; M\Delta\vec{v} \,+\, \Delta M(\vec{v} \,-\, \vec{u}) \,+\, \Delta M\Delta\vec{v}$$

Dividing by $\Delta t$

$$\vec{F}_{\rm net,\,ext} \;=\; M\frac{\Delta\vec{v}}{\Delta t} \,+\, \frac{\Delta M}{\Delta t}(\vec{v} \,-\, \vec{u}) \,+\, \frac{\Delta M}{\Delta t}\Delta\vec{v}$$

Taking limit $\Delta t \to 0$ that also means $\Delta M \to 0$ and $\Delta\vec{v} \to 0$

$$\vec{F}_{\rm net,\,ext} \;=\; M\frac{d\vec{v}}{dt} \,+\, \frac{dM}{dt}(\vec{v} \,-\, \vec{u})$$

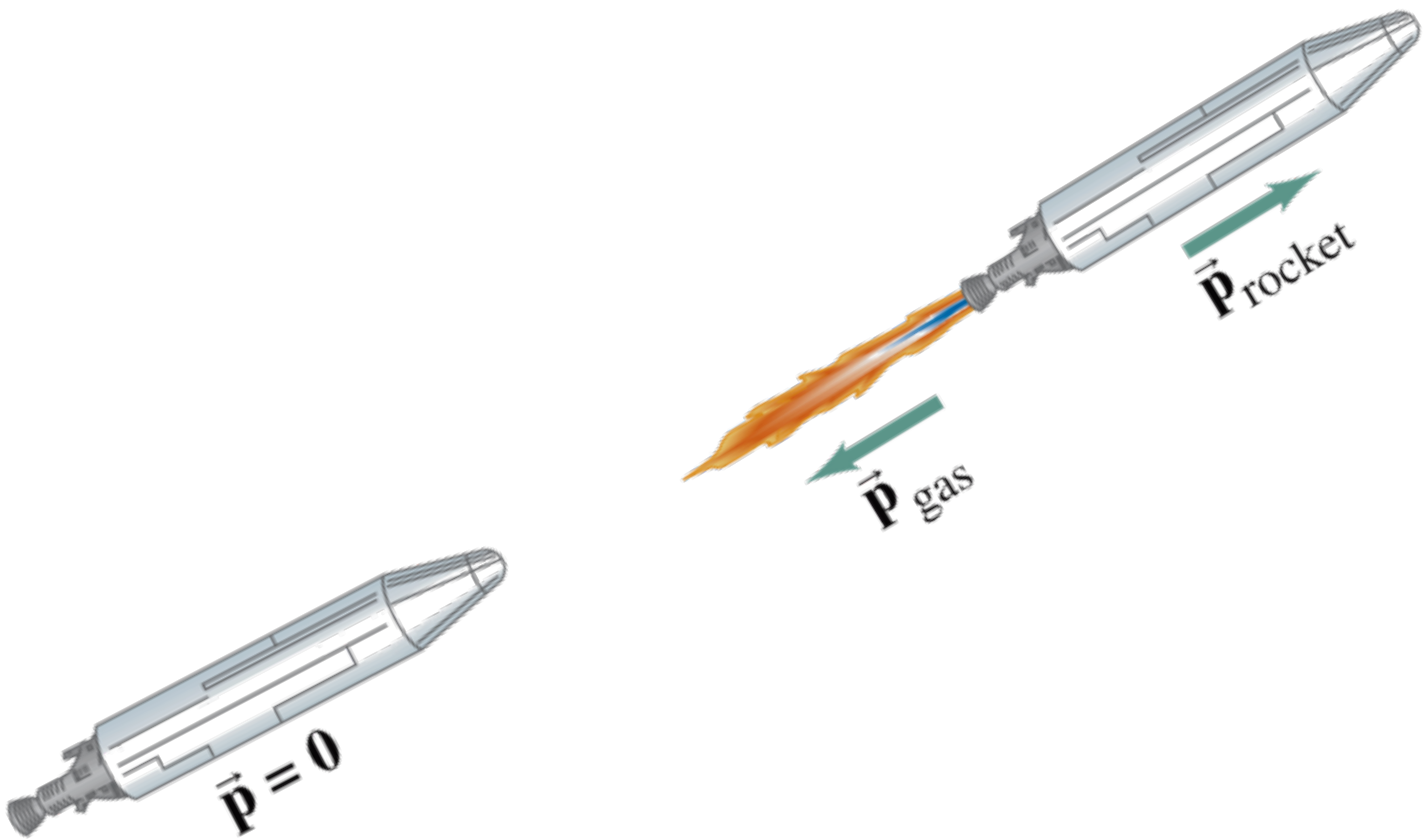
Rearranging terms we obtain Newton's second law for a system that has a continuously changing mass

$$\vec{F}_{\rm net,\,ext} \,+\, \frac{dM}{dt}\vec{v}_{\rm rel} \;=\; M\frac{d\vec{v}}{dt}$$

$$\vec{v}_{\rm rel} \;=\; \vec{u} \,-\, \vec{v}$$

# Rocket Propulsion

Momentum conservation works for a rocket as long as we consider rocket and its fuel to be one system and account for mass loss of rocket

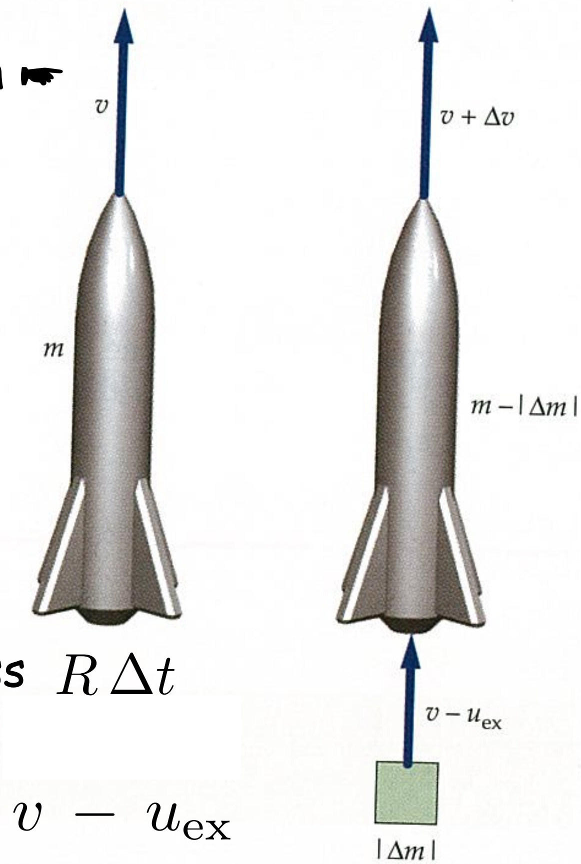


# Rocket Propulsion

Rocket propulsion is a striking example of conservation of momentum in action

Use Newton's law in form $F_{\text{ext}} = dP/dt$

Consider a rocket moving with speed $v$ relative to earth ☛




If fuel is burned at constant $R = |dm/dt|$

rocket's mass at time $t$ is ☛ $m(t) = m_0 - Rt$

Momentum of system at time $t$ is ☛ $P_{\text{i}} = mv$

At a later time $t + \Delta t$ rocket has expelled gas of mass $R\,\Delta t$

If gas is exhausted at speed $u_{\text{ex}}$ relative to rocket

velocity of gas relative to Earth is $v - u_{\text{ex}}$

Rocket then has a mass $m - R\,\Delta t$ and is moving at speed $v + \Delta v$

# Rocket Propulsion

Momentum of system at $t + \Delta t$ is

$$P_{\rm f} = (m - R\,\Delta t)(v + \Delta v) + R\,\Delta t(v - u_{\rm ex})$$
$$= mv + m\,\Delta v - v\,R\Delta t - R\,\Delta t\,\Delta v + v\,R\,\Delta t - u_{\rm ex}\,R\,\Delta t$$
$$\approx mv + m\,\Delta v - u_{\rm ex}\,R\,\Delta t$$

we dropped term $R\,\Delta t\,\Delta v$ which is product of two very small quantities

Change in momentum is

$$\Delta P = P_{\rm f} - P_{\rm i} = m\,\Delta v - u_{\rm ex}\,R\,\Delta t$$

and

$$\frac{\Delta P}{\Delta t} = m\,\frac{\Delta v}{\Delta t} - u_{\rm ex}\,R$$

As $\Delta t$ approaches zero $\Delta v/\Delta t$ approaches derivate $dv/dt$ ☛ acceleration

# Rocket Propulsion

For a rocket moving upward near surface of earth $F_{\text{ext}} = -mg$

Setting $dP/dt = F_{\text{ext}} = -mg$ gives us **rocket equation**

$$m\frac{dv}{dt} = Ru_{\text{ex}} + F_{\text{ext}} = Ru_{\text{ex}} - mg \quad \textbf{rocket equation}$$

or

$$\frac{dv}{dt} = \frac{Ru_{\text{ex}}}{m} - g = \frac{Ru_{\text{ex}}}{m_0 - Rt} - g \qquad (\ast)$$

Quantity $Ru_{\text{ex}}$ is force exerted on rocket by exhausting fuel

This is called **thrust**

$$F_{\text{th}} = Ru_{\text{ex}} = \left|\frac{dm}{dt}\right| u_{\text{ex}}$$

# Rocket Propulsion

(✶) is solved by integrating both sides with respect to time

For a rocket starting at rest at $t = 0$ result is

$$v = -u_{\rm ex} \ln \left( \frac{m_0 - Rt}{m_0} \right) - gt$$

as can be verified by taking time derivative of $v$

**Payload** of a rocket is final mass $m_{\rm f}$ after all fuel has been burned

**Burn time** $t_{\rm b}$ is given by $m_{\rm f} = m_0 - Rt_{\rm b}$ or

$$t_{\rm b} = \frac{m_0 - m_{\rm f}}{R}$$

A rocket starting at rest with mass $m_0$ and payload of $m_{\rm f}$

attains a final speed

$$v_{\rm f} = -u_{\rm ex} \ln \frac{m_{\rm f}}{m_0} - gt_{\rm b}$$ **final speed of rocket**

assuming acceleration of gravity to be constant

# Saturn V: America's Moon Rocket

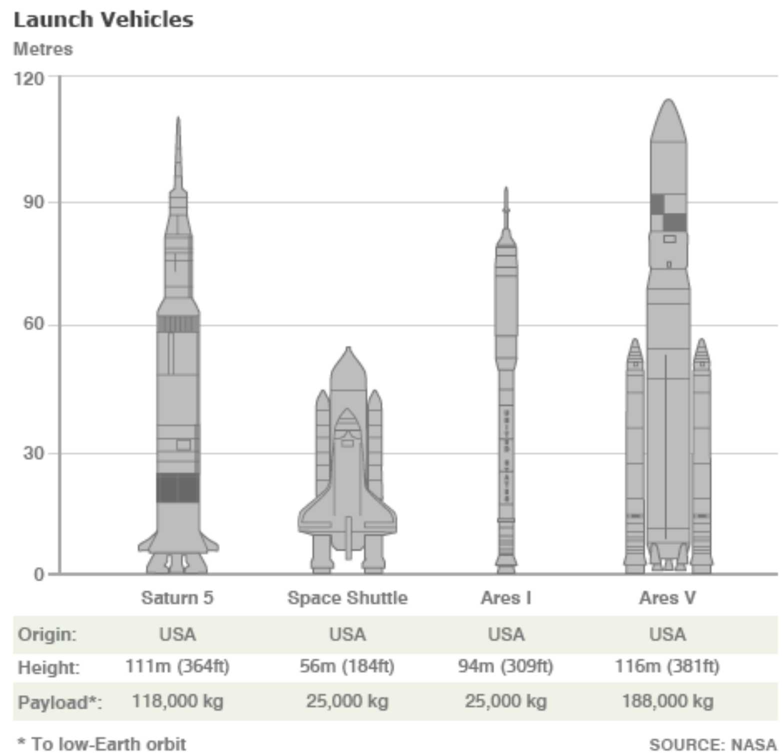
The Saturn V rocket used in Apollo moon-landing program had:

initial mass $m_0 = 2.85 \times 10^6$ kg 73% of which was fuel

a burn rate $\alpha = 13.84 \times 10^3$ kg/s

and a thrust $F_{\rm th} = 34 \times 10^6$ N

Find (a) the exhaust speed relative to the rocket; (b) the burn time; (c) the acceleration at liftoff (d) the acceleration at just before burnout; (e) the final speed of the rocket



# Saturn V: America's Moon Rocket

(a) $$F_{\rm th} = \left|\frac{dm}{dt}\right| u_{\rm ex} \Rightarrow u_{\rm ex} = 2.46 \ {\rm km/s}$$

(b) $$m_{\rm b} = 0.27 m_0 = 7.70 \times 10^5 \ {\rm kg} \qquad m_{\rm fuel} = \alpha\, t_{\rm b}$$

$$t_{\rm b} = \frac{m_{\rm fuel}}{\alpha} = \frac{m_0 - m_{\rm b}}{\alpha} = 150 \ {\rm s}$$

(c) $$\frac{dv_y}{dt} = \frac{u_{\rm ex}}{m_0}\left|\frac{dm}{dt}\right| - g = 2.14 \ {\rm m/s^2}$$

(d) $$\frac{dv_y}{dt} = \frac{u_{\rm ex}}{m_{\rm b}}\left|\frac{dm}{dt}\right| - g = 34.3 \ {\rm m/s^2}$$

(e) $$v_y = u_{\rm ex} \ln\left(\frac{m_0}{m_0 - \alpha t}\right) - gt = 1.75 \ {\rm km/s}$$

# Rotational Dynamics

# Angular Quantities

In purely rotational motion

all points on object move in circles around axis of rotation ☛ $O$

Radius of circle is $r$

All points on straight line drawn through axis move through same angle in same time

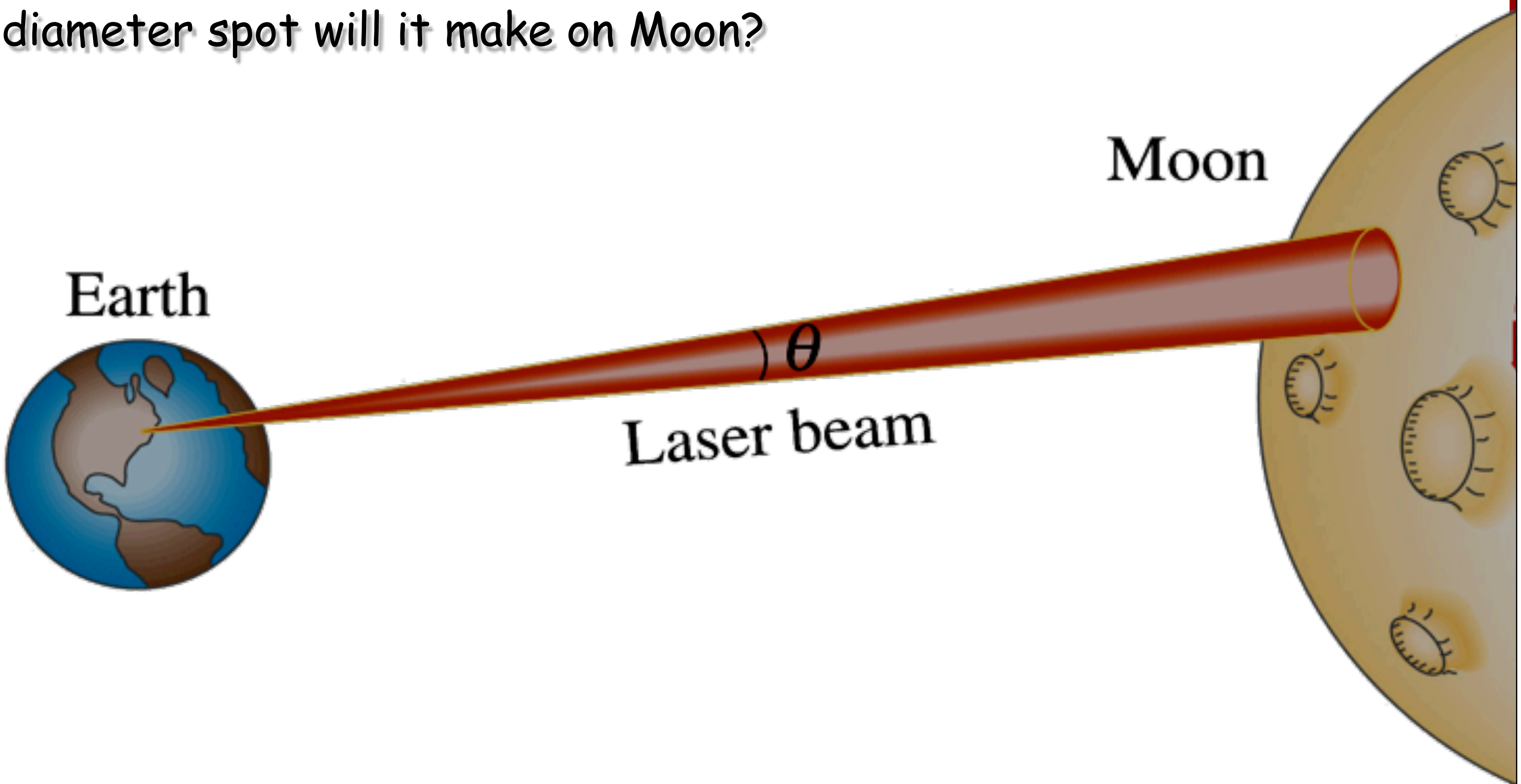
Angle $\theta$ in radians is defined ☛ $\theta = \frac{l}{r}$ → **arc length**

# Divergence of laser beam

A laser jet is directed at Moon, $380.000\, km$ from Earth

Beam diverges at an angle $\theta = 1.4 \times 10^{-5}$

What diameter spot will it make on Moon?



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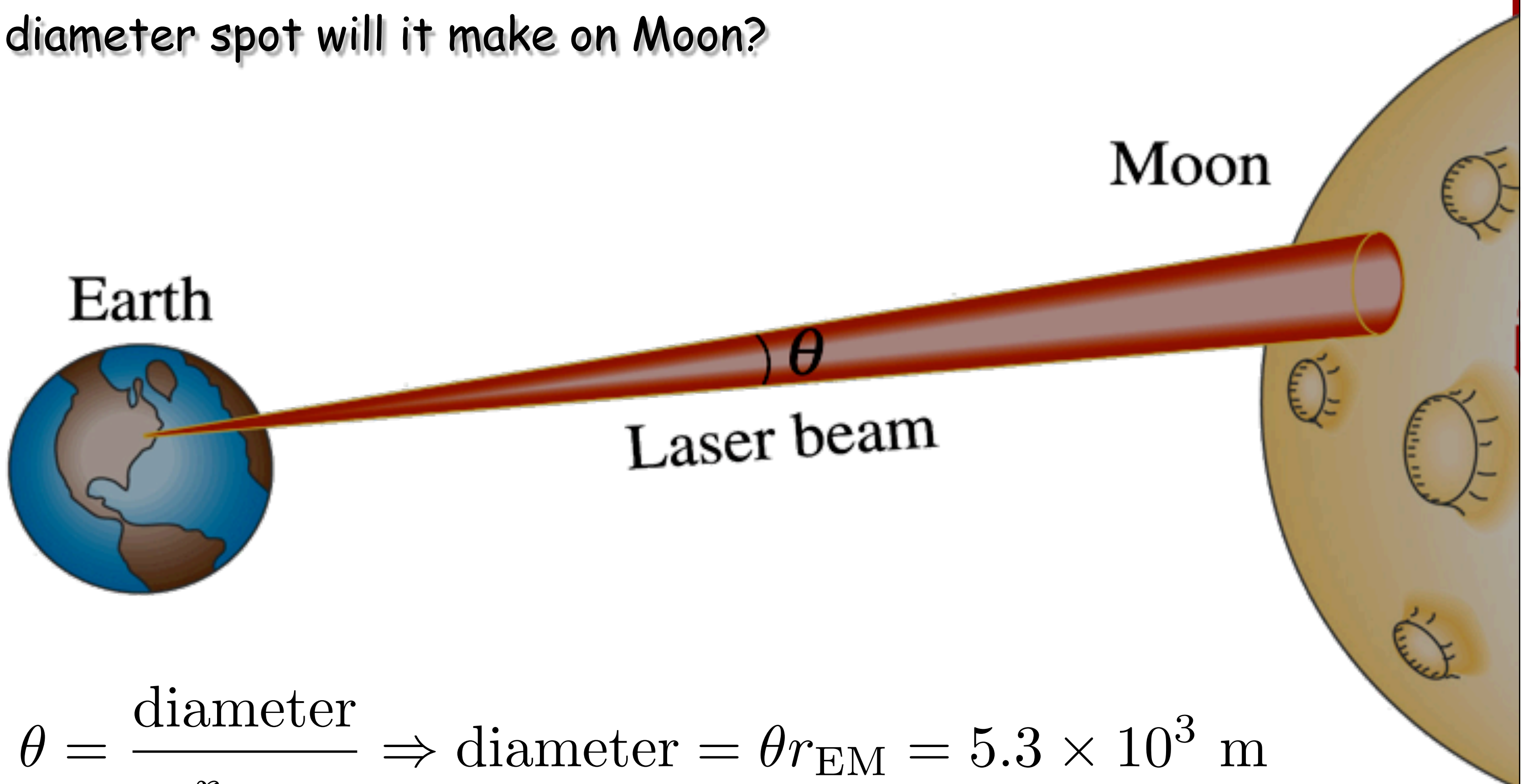
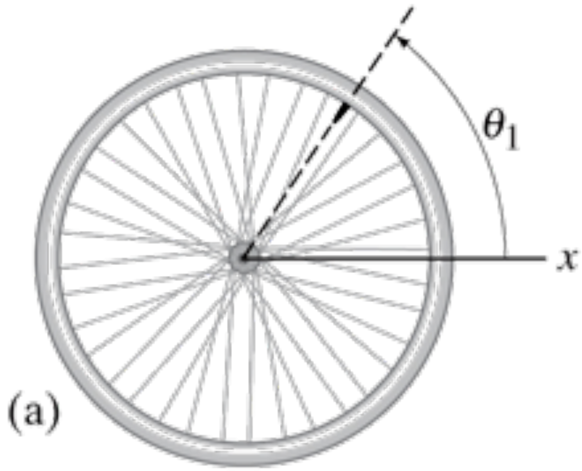



$$\theta = \frac{\text{diameter}}{r_{\text{EM}}} \Rightarrow \text{diameter} = \theta r_{\text{EM}} = 5.3 \times 10^3 \text{ m}$$

# Angular Quantities (cont'd)

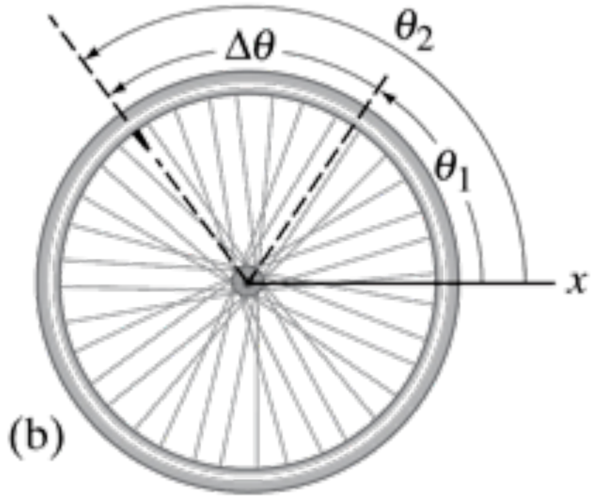



Angular displacement

$$\Delta\,\theta \;=\; \theta_2 \;-\; \theta_1$$

Average angular velocity is defined as total angular displacement divided by time

$$\bar{\omega} \;=\; \frac{\Delta\theta}{\Delta t}$$




Instantaneous angular velocity:

$$\omega \;=\; \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

# Angular Quantities (cont'd)

Angular acceleration is rate at which angular velocity changes with time

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous acceleration

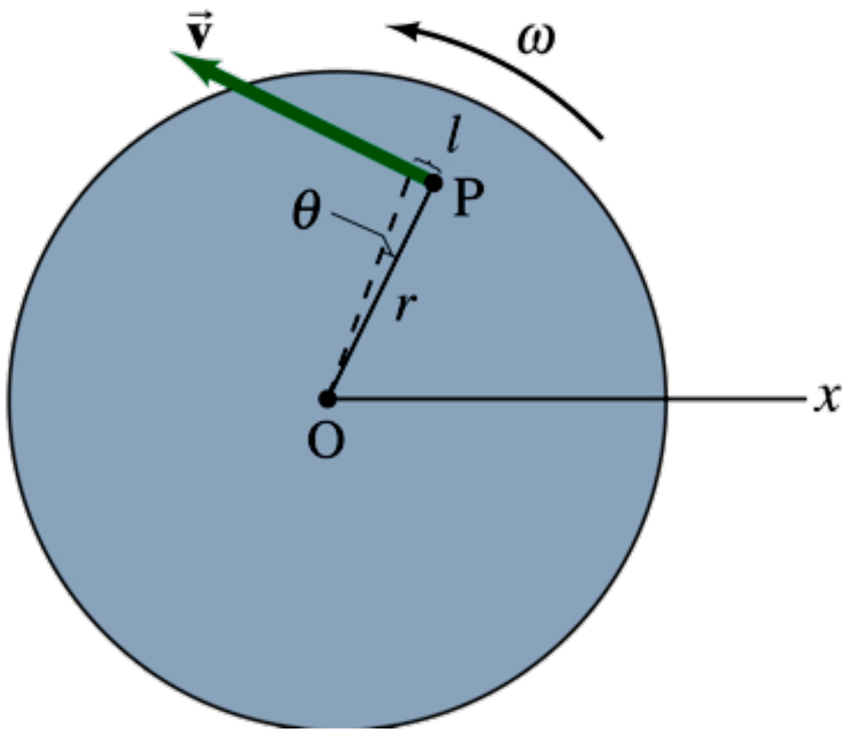
$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

# Angular Quantities

Every point on a rotating body has an angular velocity $\omega$

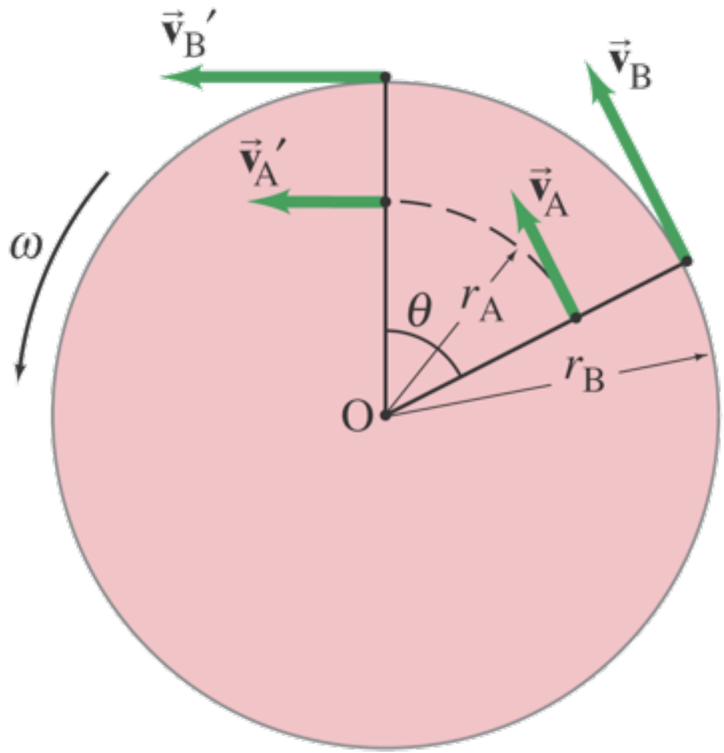
and a linear velocity $v$

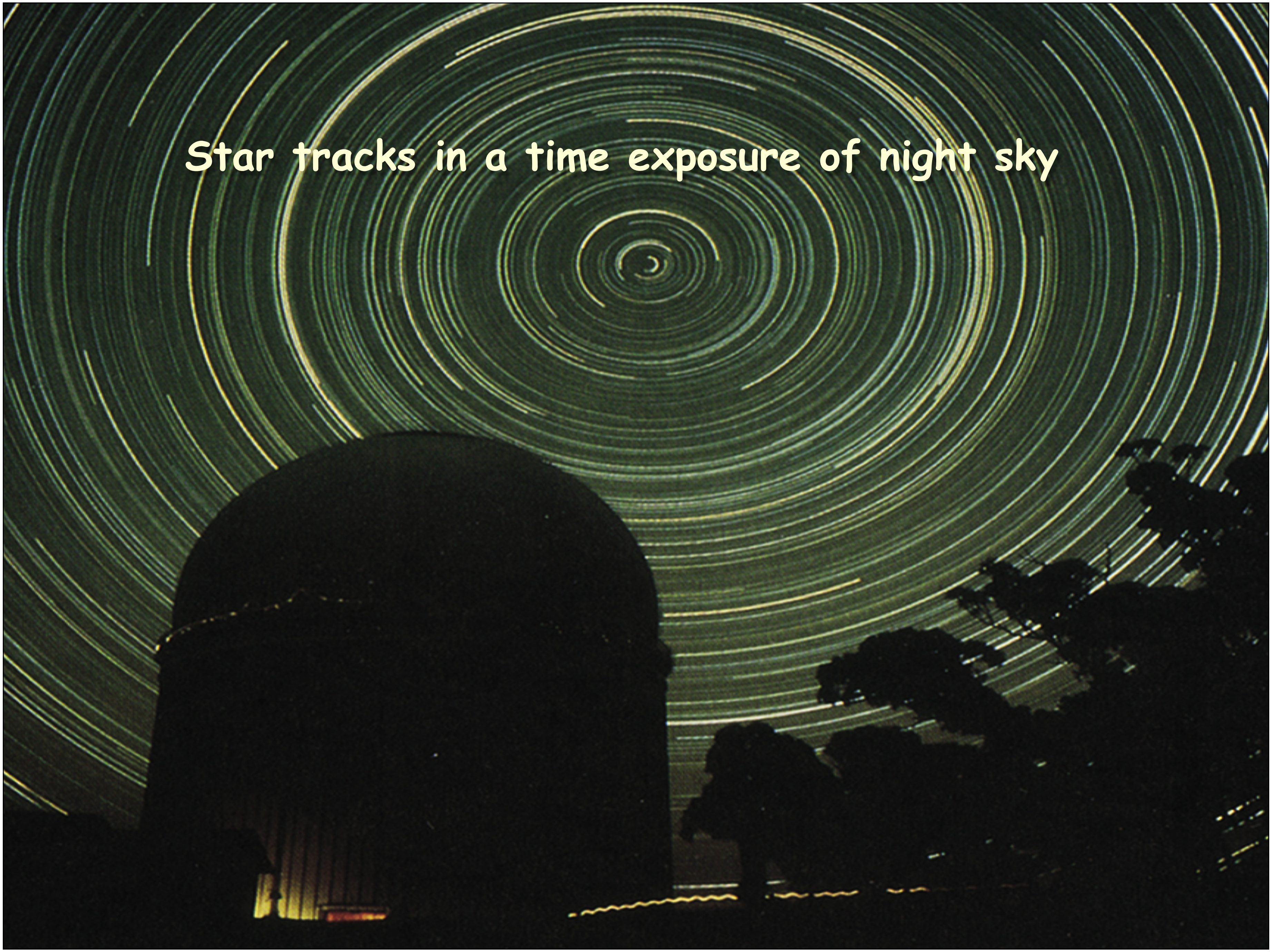
They are related ☛ $v = r\omega$



# Angular Quantities (cont'd)

**Therefore objects farther from axis of rotation will move faster**




Star tracks in a time exposure of night sky

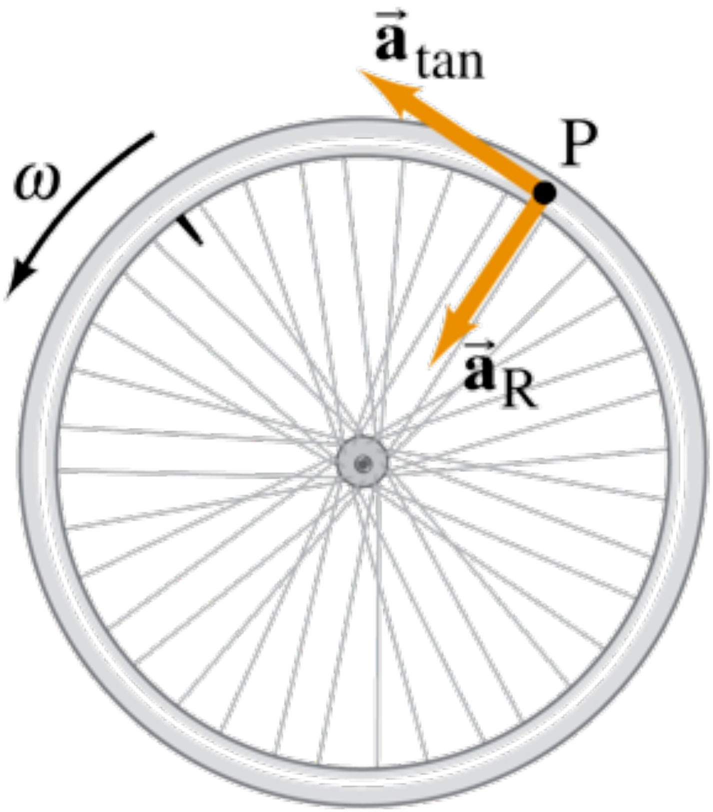
# Angular Quantities (cont'd)

If angular velocity of a rotating object changes, it has a tangential acceleration

$$a_{\tan} = r\,\alpha$$

Even if angular velocity is constant each point on object

has centripetal acceleration




$$a_R = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \omega^2\,r$$

# Angular Quantities

Here is correspondence between linear and rotational quantities

| LINEAR | TYPE | ROTATIONAL | RELATION |
|---|---|---|---|
| $x$ | DISPLACEMENT | $\theta$ | $x = r\,\theta$ |
| $v$ | VELOCITY | $\omega$ | $v = r\,\omega$ |
| $a_{\rm tan}$ | ACCELERATION | $\alpha$ | $a_{\rm tan} = r\,\alpha$ |

# Angular Quantities (cont'd)

✓ Frequency is number of complete revolutions per second

$$f = \frac{\omega}{2\pi}$$

✓ Frequencies are measured in hertz

$$1 \text{ Hz} = 1 \text{ s}^{-1}$$

✓ Period is time one revolution takes

$$T = \frac{1}{f}$$

# Rotational Kinetic Energy

Kinetic energy of rigid object rotating about fixed axis

is sum of kinetic energy of individual particles that collectively make object

Kinetic energy of the i-th particle ☛ $K = \frac{1}{2} m_i v_i^2$

Summing over all particles using $v_i = r_i \omega$ gives rotational kinetic energy

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \omega^2 \sum_i m_i r_i^2 = \frac{1}{2} I \omega^2$$

$I = \sum_i m_i r_i^2$ ☛ moment of inertia for axis of rotation

Object that has both translational and rotational motion

also has both translational and rotational kinetic energy

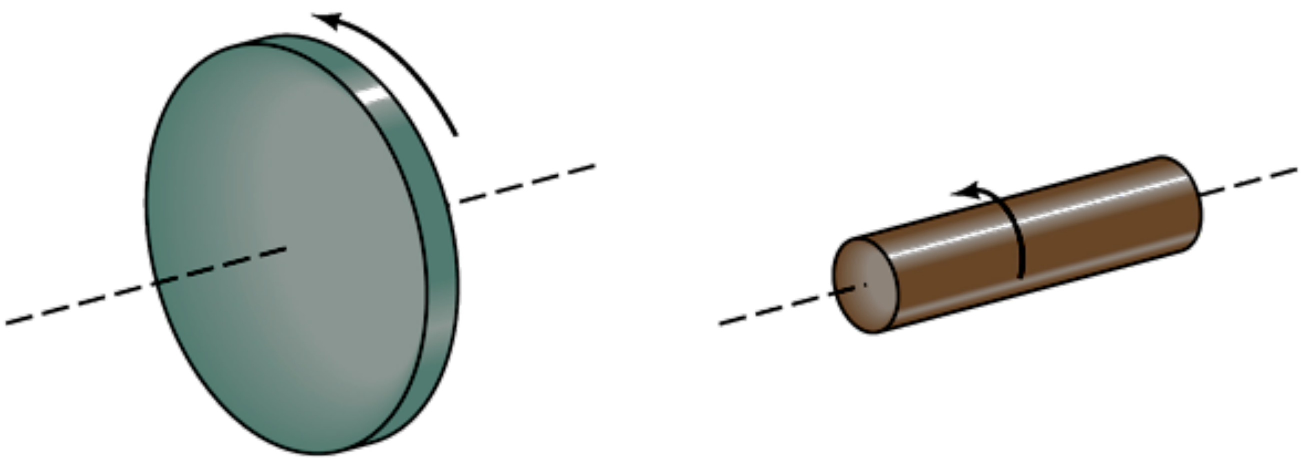
$$KE = \frac{1}{2} M v_{\rm CM}^2 + \frac{1}{2} I_{\rm CM}\, \omega^2$$

# Moment of Inertia

Quantity $I = \sum m_{\rm i}\, r_{\rm i}^2$ is called rotational inertia of an object

Distribution of mass matters here

these two objects have same mass
but one on left has a greater rotational inertia
as so much of its mass is far from axis of rotation

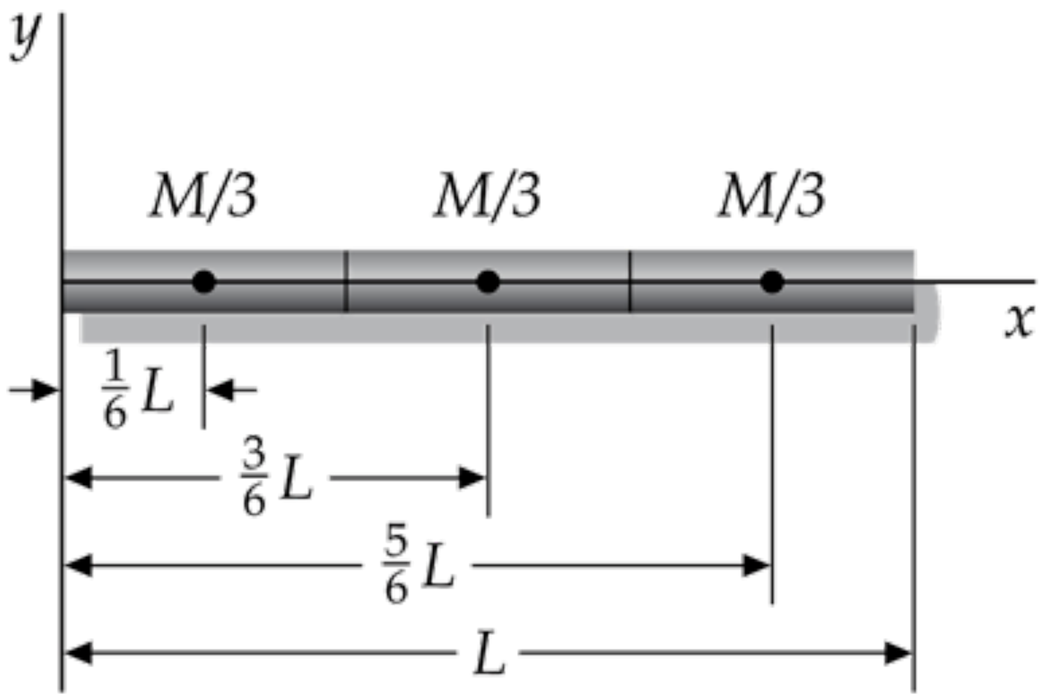


**Integral form** ☛ $I = \int r^2\, dm$

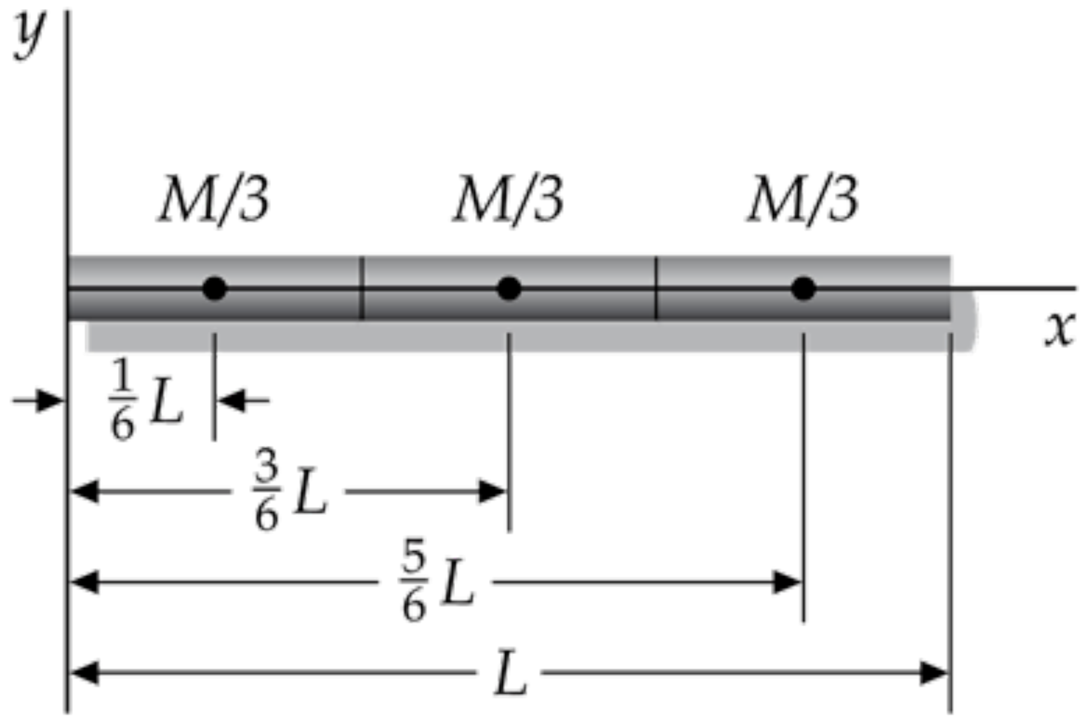
# Estimating moment of inertia

Estimate moment of inertia of a thin uniform rod of length L and mass M about an axis perpendicular to rod and through one end

Execute this estimation by modeling rod as three point masses each point mass representing 1/3 of rod



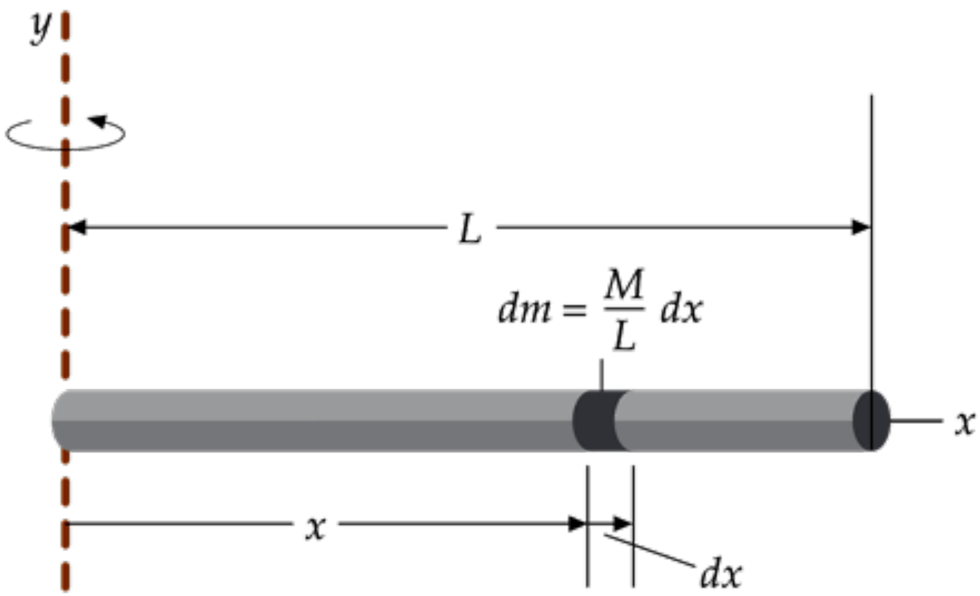
# Estimating moment of inertia



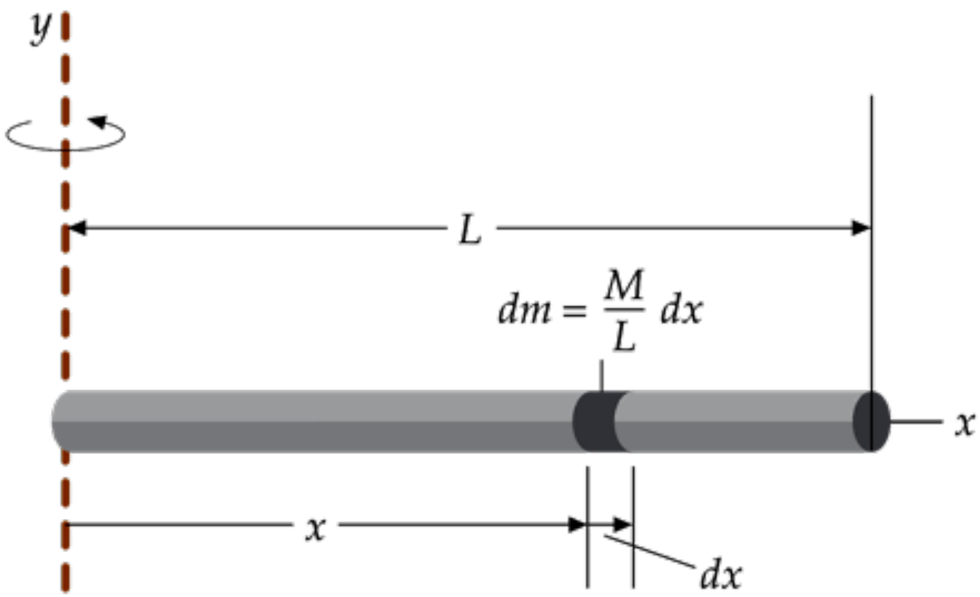

$$I = \sum m_i r_i^2 = \left(\frac{1}{3}M\right)\left(\frac{1}{6}L\right)^2 + \left(\frac{1}{3}M\right)\left(\frac{3}{6}L\right)^2 + \left(\frac{1}{3}M\right)\left(\frac{5}{6}L\right)^2 = \frac{35}{108}ML^2$$

# Moment of inertia of a thin uniform road

Find moment of inertia of a thin uniform rod of length L and mass M about an axis perpendicular to rod and through one end



# Moment of inertia of a thin uniform road



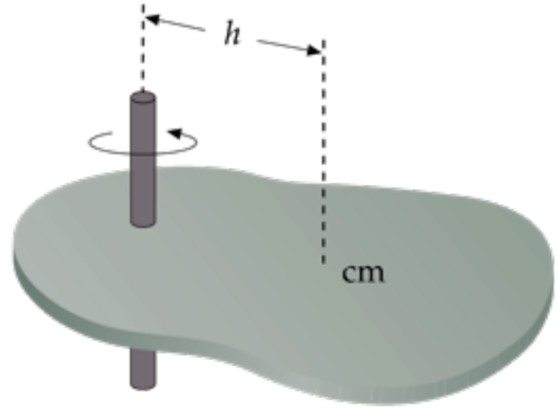

$$I = \int r^2 dm = \int x^2 dm$$

$$dm = \lambda dx = \frac{M}{L} dx$$

$$I = \int x^2 dm = \int_0^L x^2 \frac{M}{L} dx = \frac{1}{3} M L^2$$

# Steiner's Theorem

Parallel-axis theorem relates moment of inertia about an axis through CM to moment of inertia about a second parallel axis




Let $I$ be moment of inertia

Let $I_{\rm cm}$ be moment of inertia about a parallel-axis through center of mass

Let $M$ be total mass of object and $h$ distance between two axes

Parallel axis theorem states that ☛ $I = I_{\rm cm} + Mh^2$

# Steiner's Theorem (cont'd)

Consider object rotating about fixed axis that does not pass through CM

Kinetic energy of such a system is

$$K = \frac{1}{2} I\omega^2$$

Moment of inertia about fixed axis

Kinetic energy of a system can be written as

sum of its translational and rotational kinetic energy relative to its CM

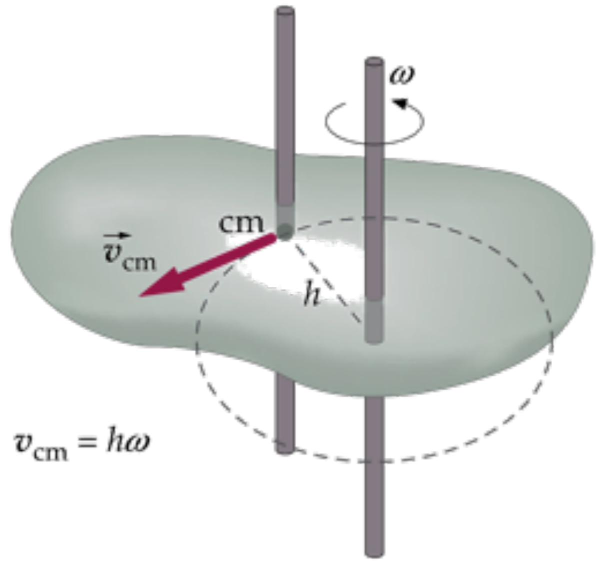
For object that is rotating relative to its CM axis

rotating kinetic energy = $\frac{1}{2} \underbrace{I_{\rm cm}} \omega^2$

Moment of inertia about axis through cm

Total kinetic energy of object is ☛ $K = \frac{1}{2} M v_{\rm cm}^2 + \frac{1}{2} I_{\rm cm} \omega^2$

# Steiner's Theorem (cont'd)




cm moves along a circular path of radius $h \rightarrow v_{\rm cm} = h\omega$

Substituting

$$\frac{1}{2}I\omega^2 = \frac{1}{2}Mh^2\omega^2 + \frac{1}{2}I_{\rm cm}\,\omega^2$$

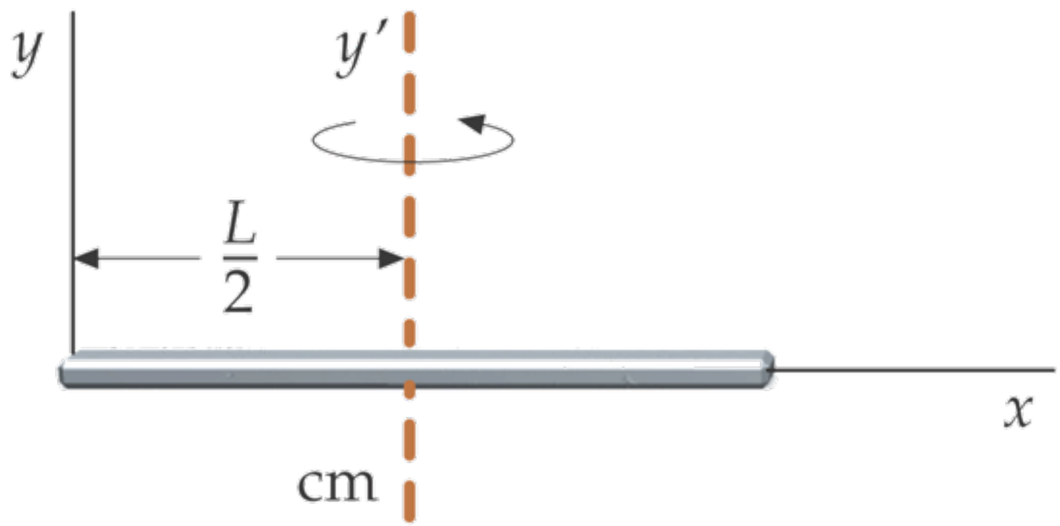
Multiplying through this equation by $2/\omega^2$ leads to

$$I = Mh^2 + I_{\rm cm}$$

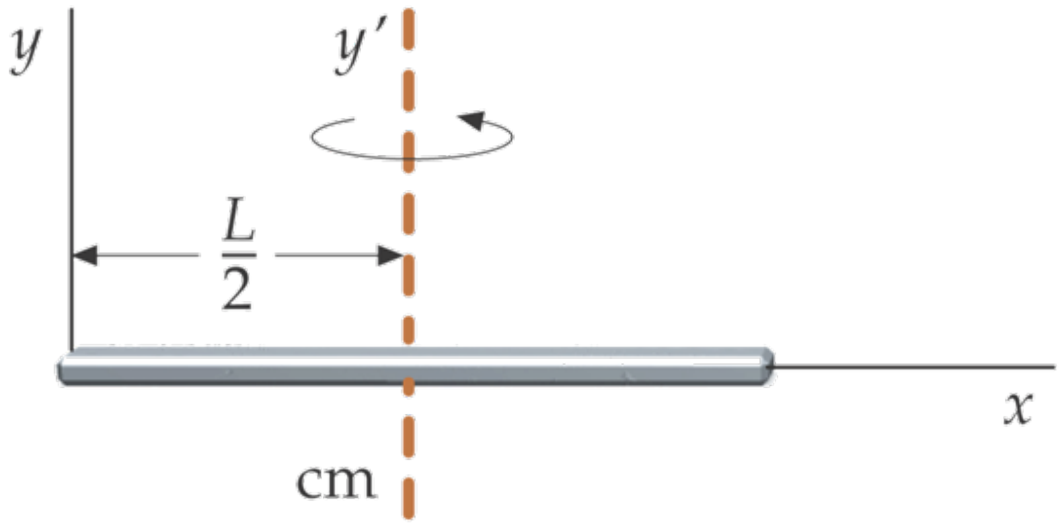
# Application of Steiner's theorem

A thin uniform rod of mass $M$ and length $L$ on $x$ axis has one end at origin

Using parallel-axis theorem, find moment of inertia about $y'$ axis, which is parallel to $y$ axis, and through center of rod



# Application of Steiner's theorem



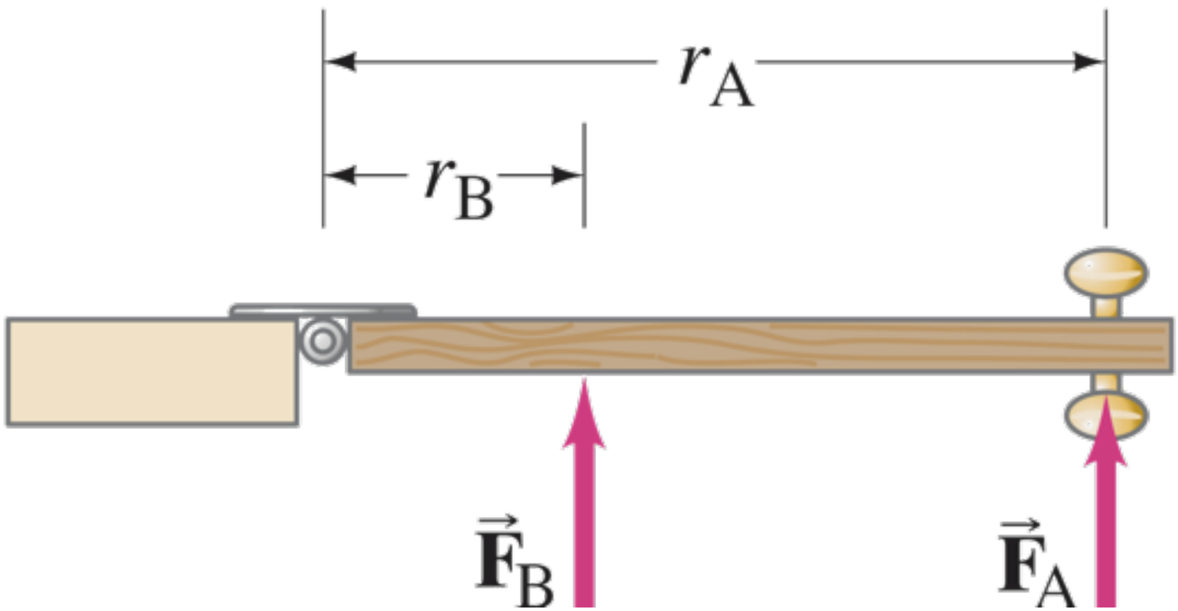

$$I_y = I_{\rm CM} + Mh^2 \Rightarrow I_{\rm CM} = I_y - Mh^2 = \frac{1}{3}ML^2 - \frac{1}{4}ML^2 = \frac{1}{12}ML^2$$

# Torque

To make an object start rotating a force is needed

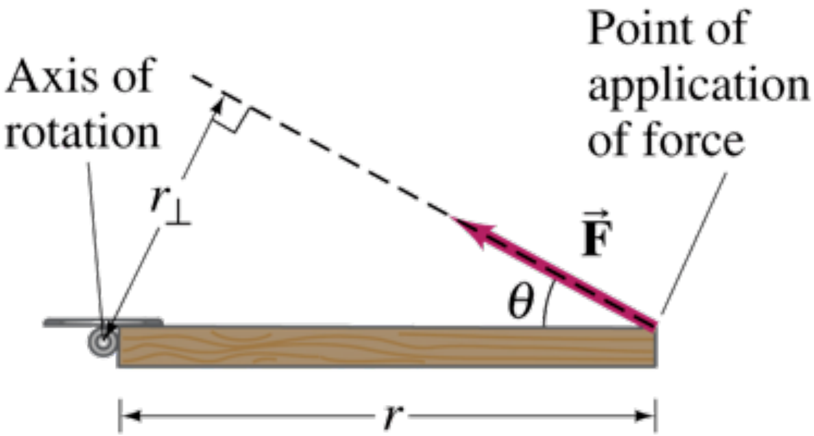
Position and direction of force matter as well

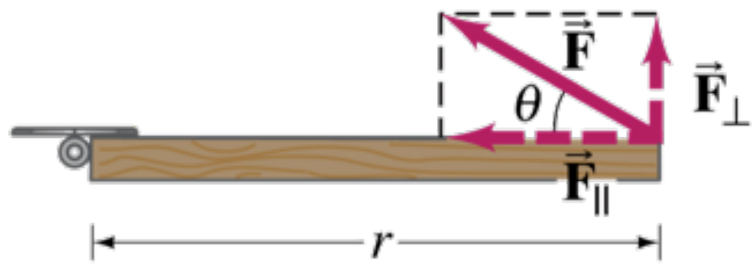
Perpendicular distance from axis of rotation
to line along which force acts is called lever arm



# Torque (Cont'd)

Torque is defined as ☛ $$\tau = r_\perp F$$



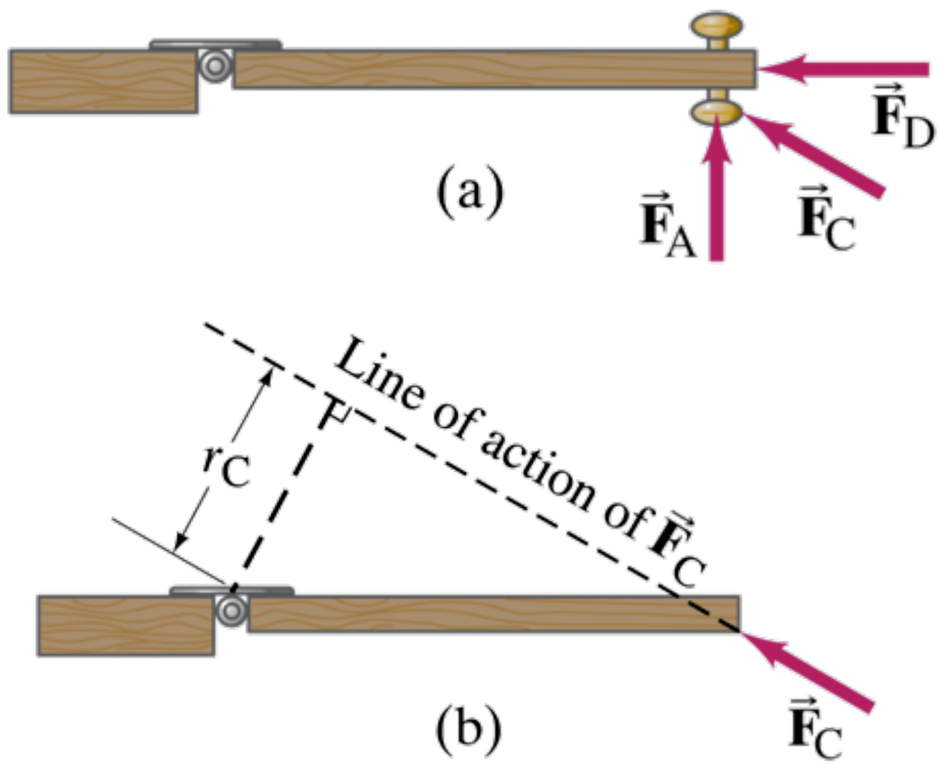



# Torque (Cont'd)

Lever arm for $F_A$ is distance from knob to hinge

Lever arm for $F_D$ is zero

Lever arm for $F_C$ is as shown

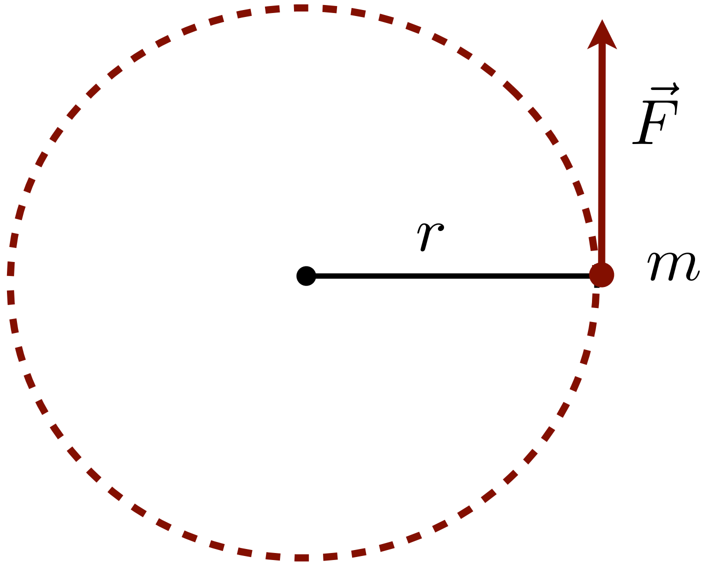


# Rotational Dynamics ☛ Torque and Rotational Inertia

Knowing that $F = ma$ ☛ $\tau = mr^2\,\alpha$

This is for a single point mass

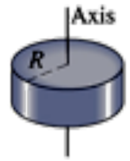
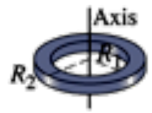
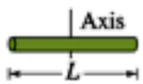
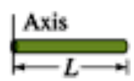
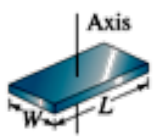
What about an extended object?




As angular acceleration is same for whole object we can write

$$\sum \tau_{\mathrm{i,\,net}} = \left(\sum m_{\mathrm{i}}\, r_{\mathrm{i}}^2\right) \alpha$$

# Various Moment of Inertia

| | Object | Location of axis | | Moment of inertia |
|---|---|---|---|---|
| (a) | **Thin hoop,** radius $R$ | Through center |   | $MR^2$ |
| (b) | **Thin hoop,** radius $R$ width $W$ | Through central diameter |   | $\frac{1}{2}MR^2 + \frac{1}{12}MW^2$ |
| (c) | **Solid cylinder,** radius $R$ | Through center |   | $\frac{1}{2}MR^2$ |
| (d) | **Hollow cylinder,** inner radius $R_1$ outer radius $R_2$ | Through center |   | $\frac{1}{2}M(R_1^2 + R_2^2)$ |
| (e) | **Uniform sphere,** radius $R$ | Through center |   | $\frac{2}{5}MR^2$ |
| (f) | **Long uniform rod,** length $L$ | Through center |   | $\frac{1}{12}ML^2$ |
| (g) | **Long uniform rod,** length $L$ | Through end |   | $\frac{1}{3}ML^2$ |
| (h) | **Rectangular thin plate,** length $L$, width $W$ | Through center |   | $\frac{1}{12}M(L^2 + W^2)$ |

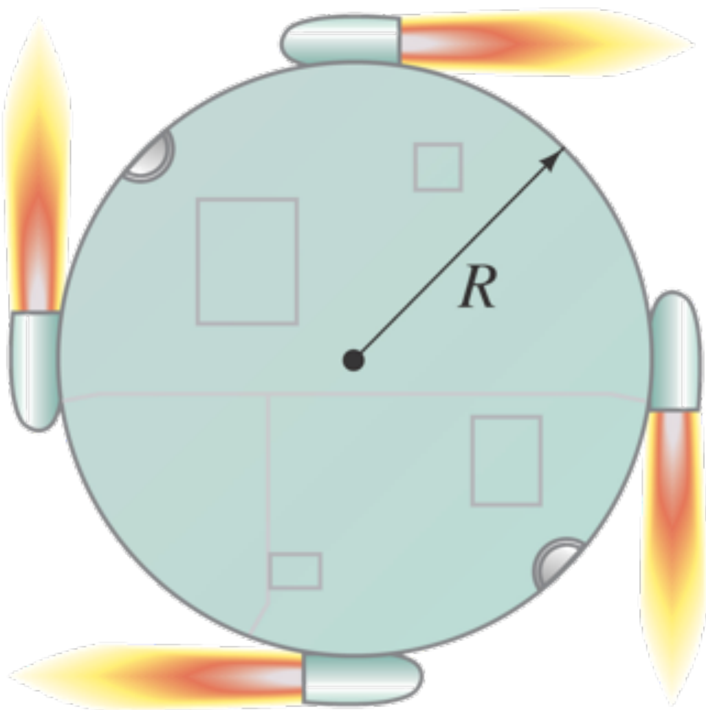
Rotational inertia of an object depends not only on its mass distribution but also location of axis of rotation

– compare (f) and (g), for example -

# Spinning Cylindrical Satellite

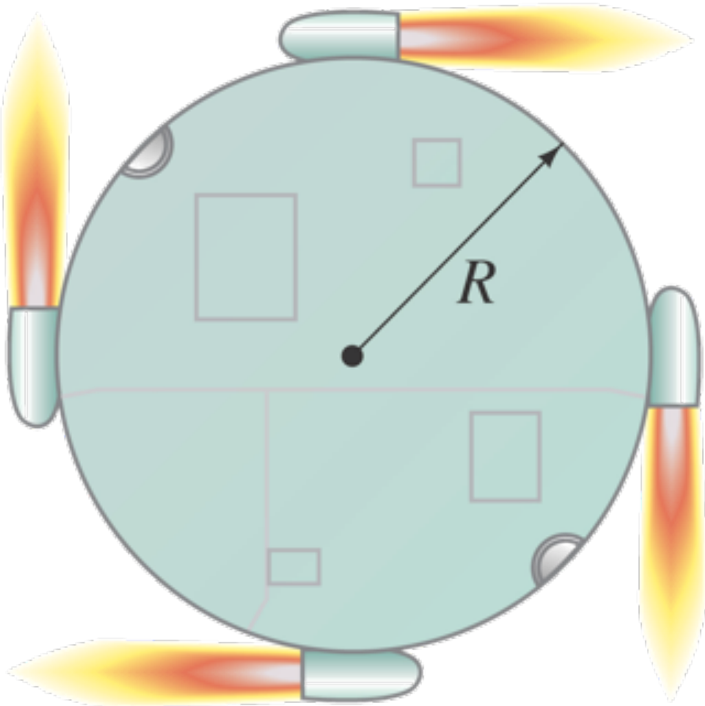
To get a flat, uniform cylindrical satellite spinning at correct rate, engineers fire four tangential rockets as shown in figure

If satellite has a mass of 3600 kg and a radius of 4 m, what is required steady force of each rocket if satellite is to reach 32 rpm in 5 min?




**End view of cylindrical satellite**

## The ring of rockets will create a torque with zero net force




$$\tau_{\rm net} = 4FR$$

$$\tau_{\rm net} = I\alpha$$

$$I = \frac{1}{2}MR^2$$

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

$$4FR = I\alpha = \frac{1}{2}MR^2\frac{\Delta\omega}{\Delta t} \Rightarrow F = \frac{1}{8}MR\frac{\Delta\omega}{\Delta t} \approx 20\ {\rm N}$$