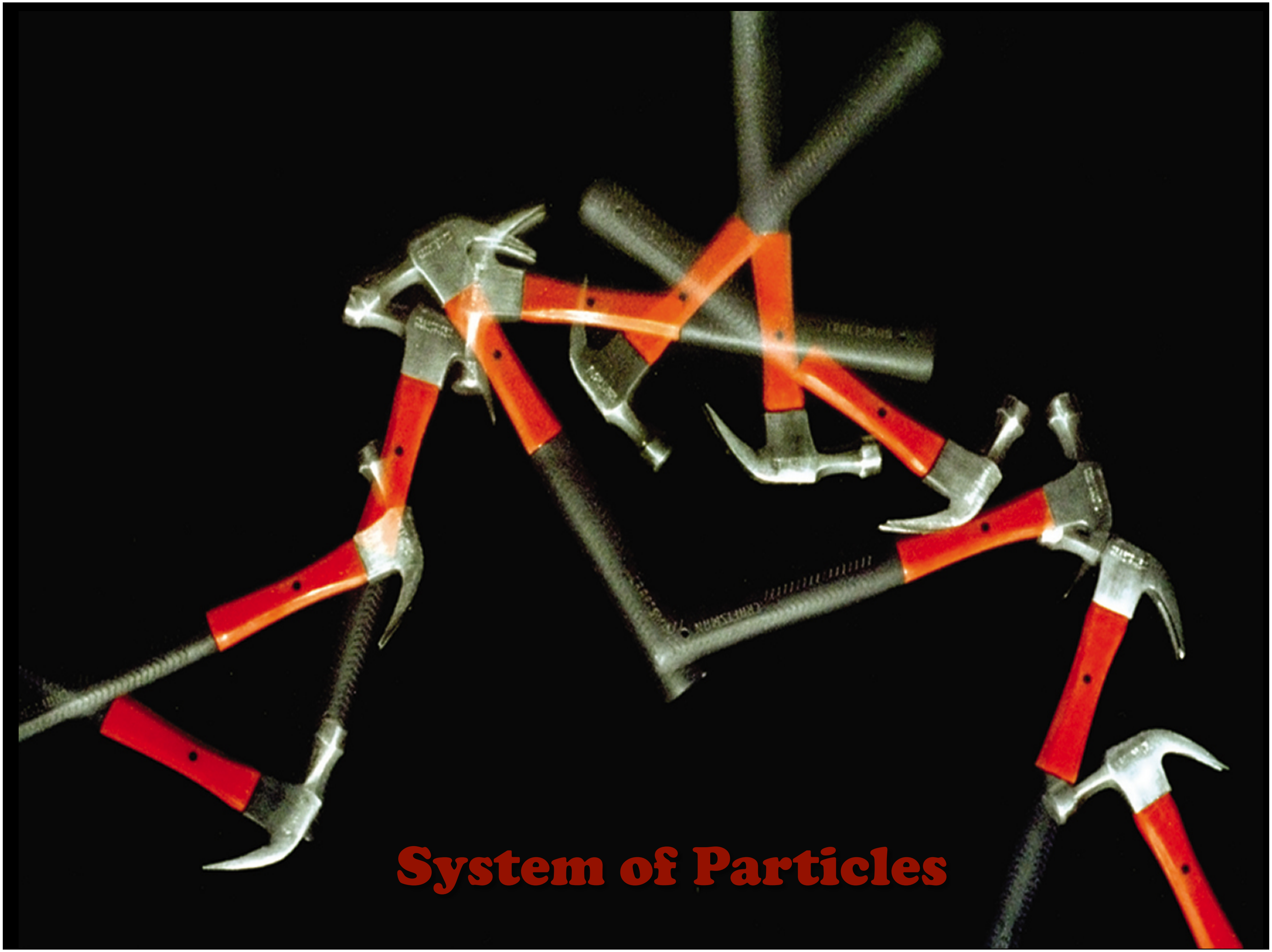
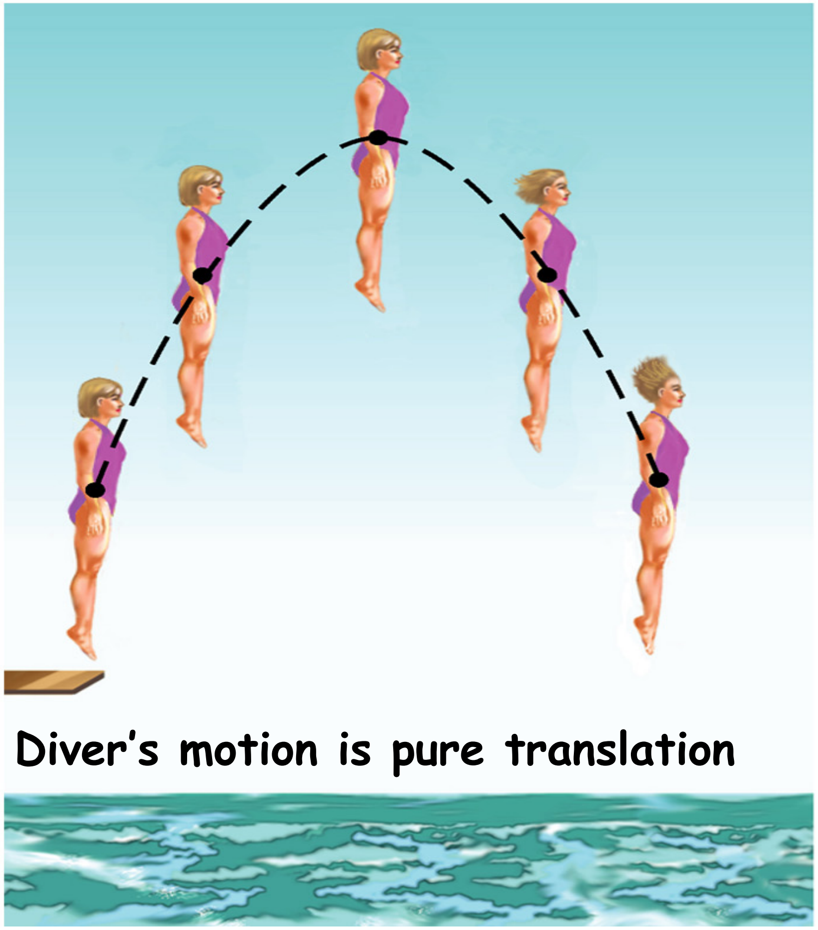
Lesson 6

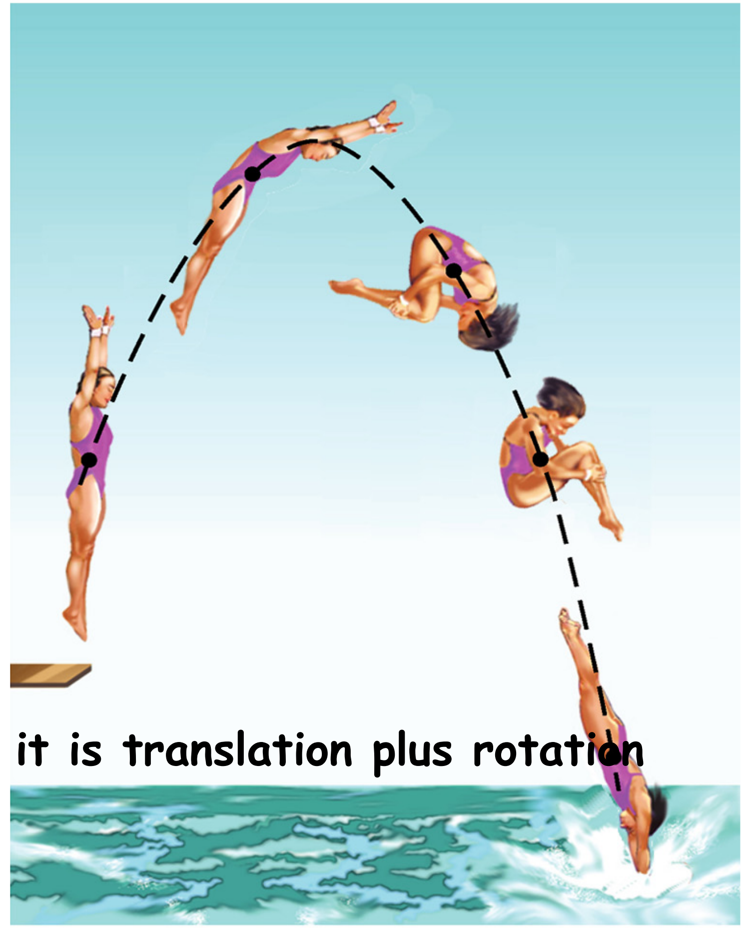
Physics 168

Luis Anchordoqui


System of Particles

Center of Mass





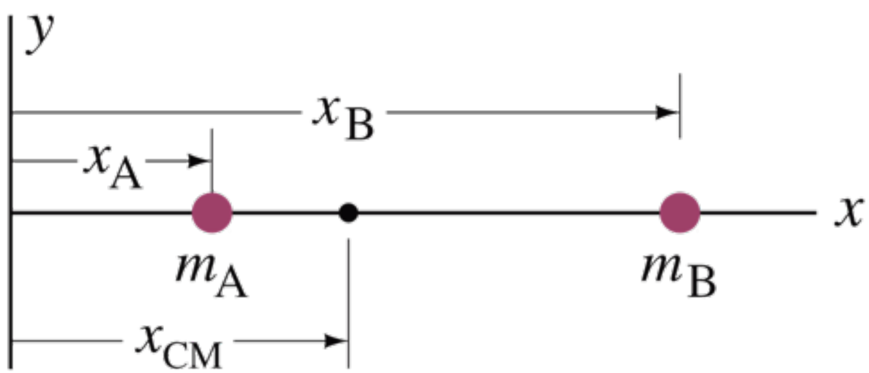
There is one point that moves in same path
a particle would take if subjected to same force as diver
This point is called center of mass (CM)

Center of Mass

For two particles center of mass reads

$$x_{CM} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = \frac{m_A x_A + m_B x_B}{M}$$

Where M is Total Mass



Center of Mass

We can generalize from two particles in one dimension

to a system of N particles in three dimensions

$$M\, x_{cm} = \sum_i^N m_i\, x_i$$

$$M\, y_{cm} = \sum_i^N m_i\, y_i$$

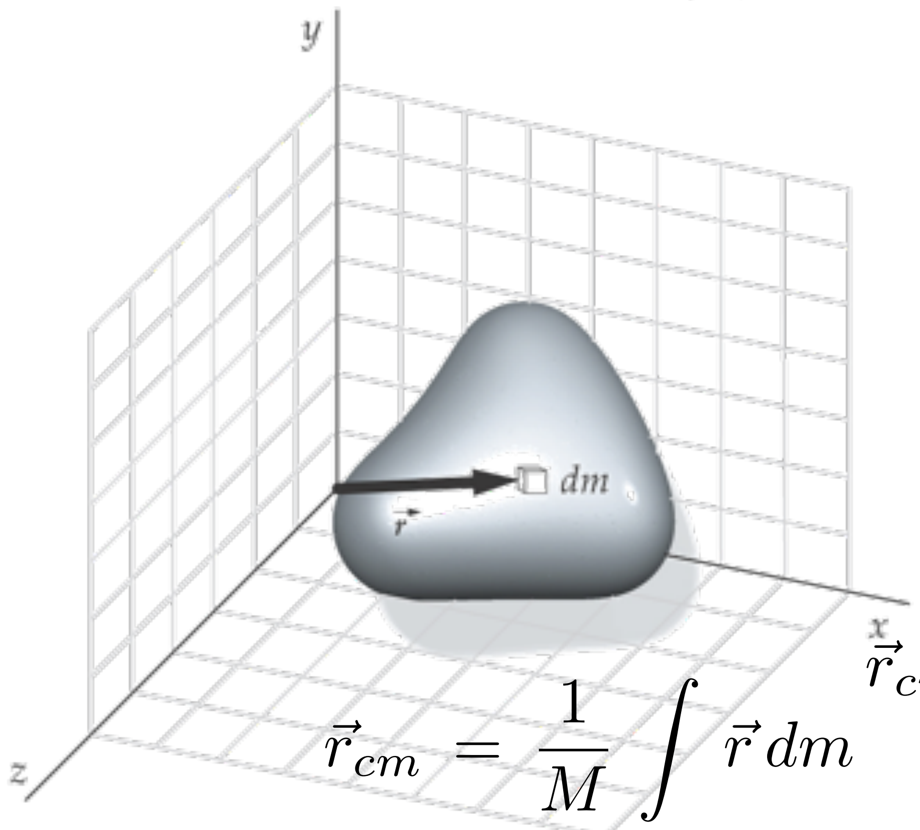
$$M\, z_{cm} = \sum_i^N m_i\, z_i$$

$$M = \sum_i^N m_i$$

$$M\, \vec{r}_{cm} = \sum_i^N m_i\, \vec{r}_i$$

$$\vec{r}_{cm} = x_{cm}\, \hat{\imath} + y_{cm}\, \hat{\jmath} + z_{cm}\, \hat{k}$$

For a continuous distribution of mass



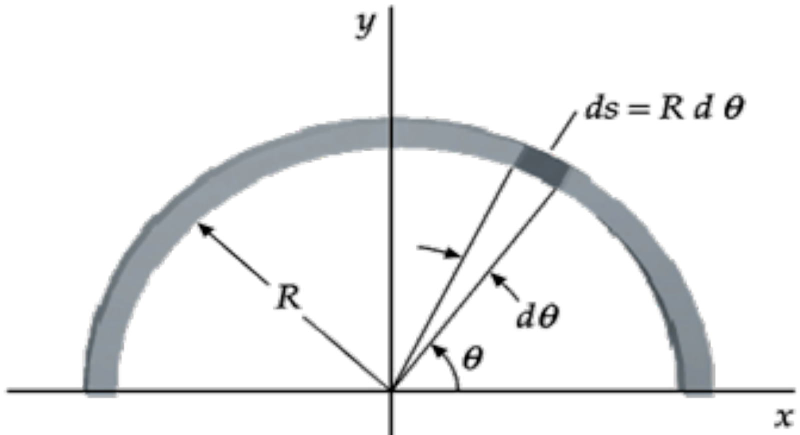
$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r}\, dm$$

Center of Mass of a uniform semicircular hoop

Use polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

Distance of points on semicircle from origin is $r = R$



$$\vec{r}_{cm} = \frac{1}{M}\int (x\hat{\imath} + y\hat{\jmath})\, dm = \frac{1}{M}\int R(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})dm$$

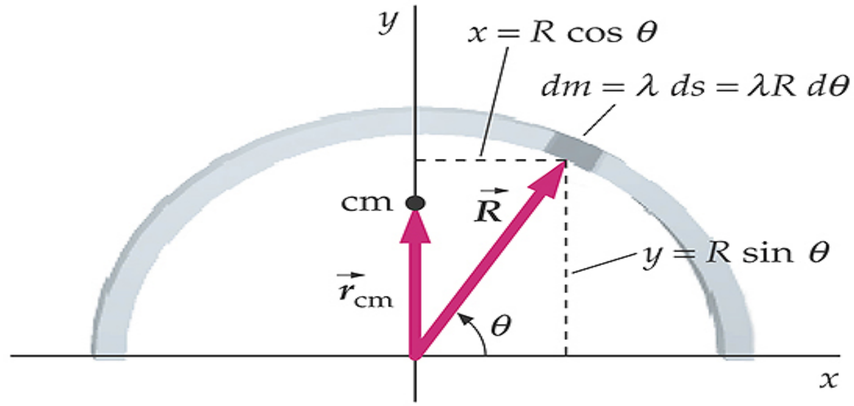
mass per unit length is $\lambda = dm/ds$

$$dm = \lambda\, ds = \lambda R\, d\theta$$

$$\vec{r}_{cm} = \frac{1}{M}\int_0^{\pi} R(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})\lambda\, R\, d\theta$$

Center of Mass of a uniform semicircular hoop (cont'd)

$$\vec{r}_{cm} = \frac{\lambda R^2}{M}\left(\hat{\imath}\int_0^\pi \cos\theta\, d\theta + \hat{\jmath}\int_0^\pi \sin\theta\, d\theta\right)$$



$$\vec{r}_{cm} = \frac{R}{\pi}\left(\hat{\imath}\int_0^\pi \cos\theta\, d\theta + \hat{\jmath}\int_0^\pi \sin\theta\, d\theta\right)$$

$$= \frac{R}{\pi}\left(\hat{\imath}\sin\theta\Big|_0^\pi - \hat{\jmath}\cos\theta\Big|_0^\pi\right)$$

$$= \frac{2}{\pi}R\hat{\jmath}$$

Curiously it is outside material of semicircular hoop!

Center of Mass for human body

High jumpers have developed a technique where their CM actually passes under bar as they go over it

This allows them to clear higher bars

Center of Mass & Translational Motion

- ☛ Sum of all forces acting on a system is equal to total mass of system multiplied by acceleration of center-of-mass

- ☛ For each internal force acting on a particle in system there is an equal and opposite internal force acting on some other particle of system

$$M\vec{a}_{CM} = \vec{F}_{net,ext}$$

Changing Places in a Rowboat

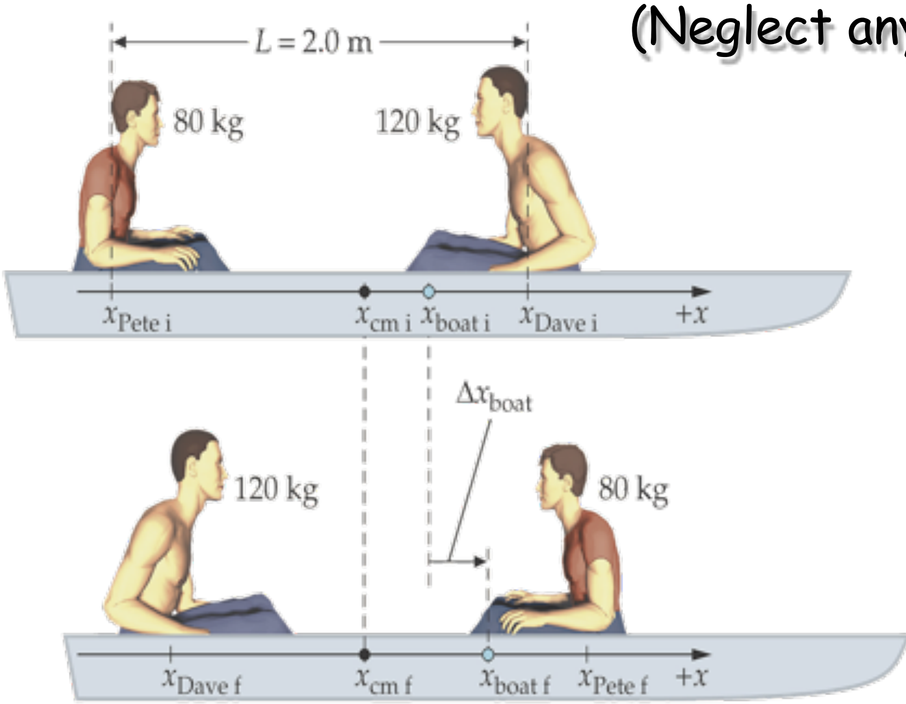
Pete (mass 80 kg) and Dave (mass 120 kg) are in a rowboat (mass 60 kg) on a calm lake

Dave is near bow of boat, rowing, and Pete is at stern, 2 m from Dave

Dave gets tired and stops rowing

Pete offers to row, so after boat comes to rest they change places

How far does boat moves as Pete and Dave change places



(Neglect any horizontal force exerted by water)

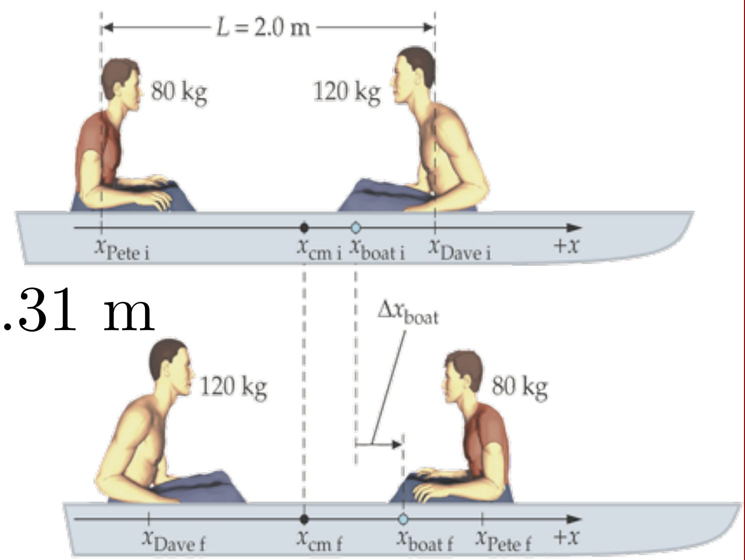
$$MX_{\mathrm{CM}_i} = m_{\mathrm{Pete}}\ x_{\mathrm{Pete}_i} + m_{\mathrm{Dave}}\ x_{\mathrm{Dave}_i} + m_{\mathrm{boat}}\ x_{\mathrm{boat}_i}$$

$$MX_{\mathrm{CM}_f} = m_{\mathrm{Pete}}\ x_{\mathrm{Pete}_f} + m_{\mathrm{Dave}}\ x_{\mathrm{Dave}_f} + m_{\mathrm{boat}}\ x_{\mathrm{boat}_f}$$

$$M\Delta X_{\mathrm{CM}} = m_{\mathrm{Pete}}\ \Delta x_{\mathrm{Pete}} + m_{\mathrm{Dave}}\ \Delta x_{\mathrm{Dave}} + m_{\mathrm{boat}}\ \Delta x_{\mathrm{boat}}$$

$$Ma_{\mathrm{CM}} = F_{\mathrm{ext,net}} = \sum_i F_{\mathrm{ext}_i} = 0$$

$$0 = m_{\mathrm{Pete}}(\Delta x_{\mathrm{boat}} + L) + m_{\mathrm{Dave}}(\Delta x_{\mathrm{boat}} - L) + m_{\mathrm{boat}}\Delta x_{\mathrm{boat}}$$



$$\Delta x_{\mathrm{boat}} = \frac{L\ (m_{\mathrm{Dave}} - m_{\mathrm{Pete}})}{m_{\mathrm{Dave}} + m_{\mathrm{Pete}} + m_{\mathrm{boat}}} = 0.31\ \mathrm{m}$$

Center of Mass Work

For a system of particles we have

$$\vec{F}_{\rm net,ext} = \sum \vec{F}_{\rm iext} = M\vec{a}_{\rm cm}$$

$$M = \sum_i m_i$$

$$\vec{F}_{\rm net,ext} \cdot \vec{v}_{\rm cm} = M\vec{a}_{\rm cm} \cdot \vec{v}_{\rm cm} = \frac{d}{dt}\left(\frac{1}{2}M\, v_{\rm cm}^2\right) = \frac{dK_{\rm trans}}{dt}$$

Center of mass work - translational kinetic energy relation

$$\int_1^2 \vec{F}_{\rm net,ext} \cdot d\vec{l}_{cm} = \Delta K_{\rm trans}$$

Power

Power is rate at which work is done

$$\bar{P} = \text{average power} = \frac{\text{work}}{\text{time}} = \frac{\text{energy transformed}}{\text{time}}$$

In SI system, units of power are watts:

$$1W = 1J/s$$

Difference between walking and running up these stairs is power

- change in gravitational potential energy is same-

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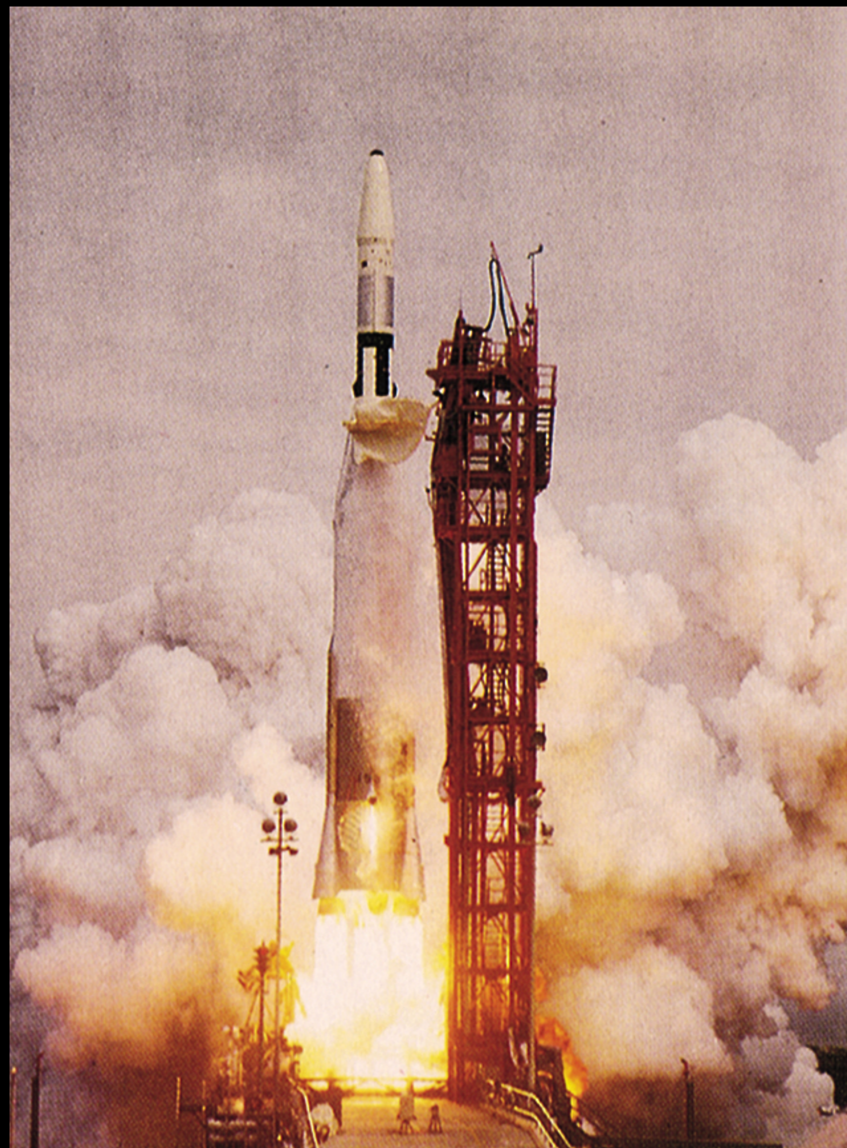
Conservation Theorems: Momentum



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Momentum of a particle

Originally introduced by Newton as quantity of motion

momentum is defined as ☚ $\vec{p} = m\vec{v}$

Using Newton´s second law we can relate

momentum of a particle to force acting on particle

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Substituting force $\vec{F}_{\rm net}$ by

$$\vec{F}_{\rm net} = \frac{d\vec{p}}{dt}$$

Net force acting on a particle equals time rate of change of particle´s momentum

In his famous treatise Principia (1687) Newton presents

second law of motion in this form

Conservation of Momentum

Total momentum of a system of particles reads

$$\vec{P}_{\rm sys} = \sum_i m_i\,\vec{v}_i = \sum_i \vec{p}_i$$

$$\vec{P}_{\rm sys} = \sum_i m_i\,\vec{v}_i = M\vec{v}_{\rm cm}$$

$$M = \sum_i m_i$$

$$\frac{d\vec{P}_{\rm sys}}{dt} = M\frac{d\vec{v}_{\rm cm}}{dt} = M a_{\rm cm}$$

According to Newton´s second law

$$\sum_i \vec{F}_{\rm ext} = \vec{F}_{\rm net,\,ext} = \frac{d\vec{P}_{\rm sys}}{dt}$$

If $\sum \vec{F}_{\rm ext} = 0$ then $\vec{P}_{\rm sys} = M\,\vec{v}_{\rm cm} = \text{constant}$

If sum of external forces on a system remains zero total momentum of system is conserved

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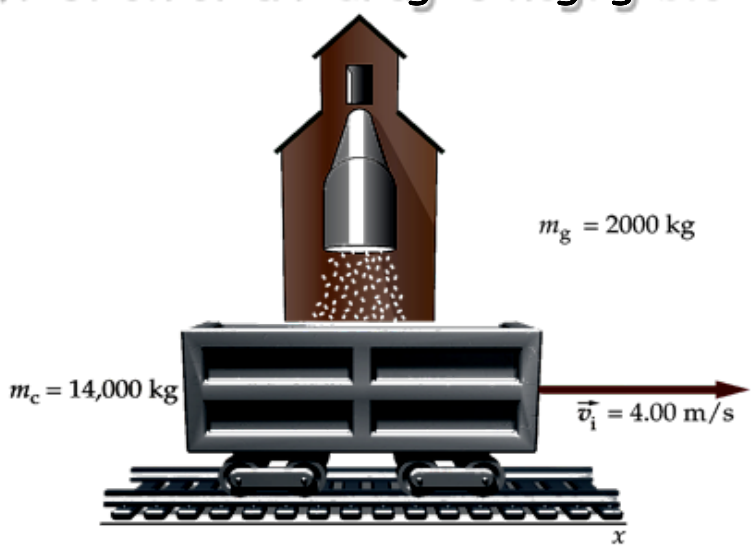
A runaway railroad car

A runaway 14,000 kg railroad car is rolling horizontally at 4 m/s toward a switchyard

As it passes by a grain elevator 2000 kg of grain suddenly drops into the car

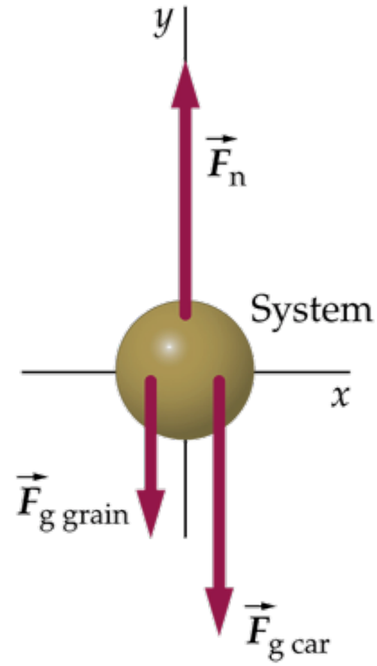
How long does it take car to cover 500 m distance from elevator to switchyard?

Assume that grains falls straight down and that slowing due to rolling friction or air drag is negligible



A runaway railroad car

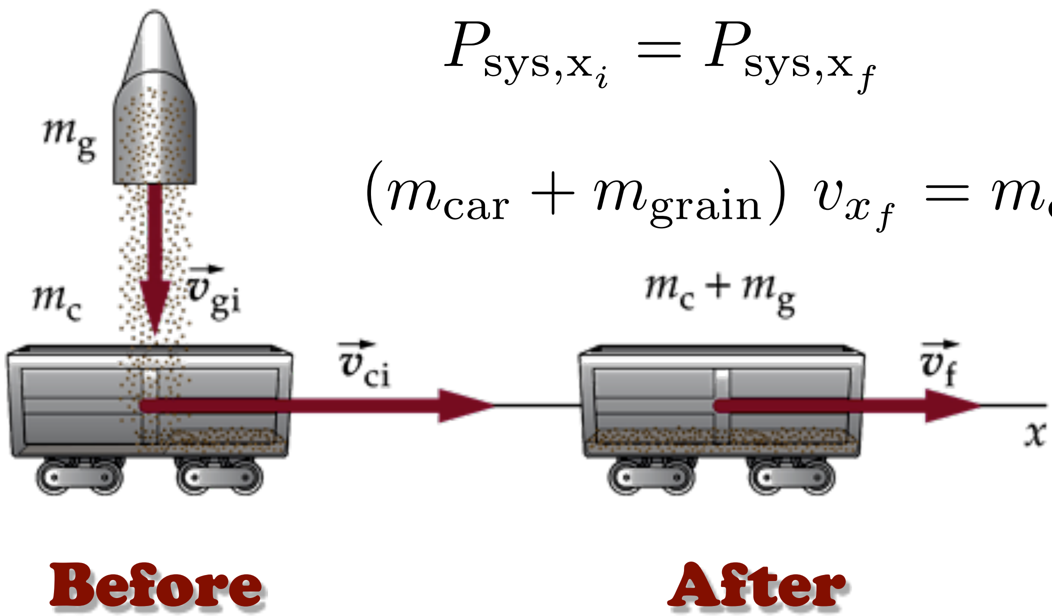
$$\sum_i \vec{F}_{\text{ext}_i} = \vec{F}_{g_{\text{grain}}} + \vec{F}_{g_{\text{car}}} + \vec{F}_n = \frac{d\vec{P}_{\text{sys}}}{dt}$$



x component of net external force is zero

$$P_{\text{sys,x}_i} = P_{\text{sys,x}_f}$$

$$(m_{\text{car}} + m_{\text{grain}})\; v_{x_f} = m_{\text{car}}\; v_{x_i}$$



$$v_{x_f} = \frac{m_{\text{car}}\; v_{x_i}}{m_{\text{car}} + m_{\text{grain}}} \Rightarrow \Delta t = \frac{d}{v_{x_f}} = 143 \text{ s}$$

Collisions and impulse

When two objects collide

they usually exert very large forces on each other for very brief time

Impulse of a force exerted during a time $\Delta t = t_f - t_i$ is a vector defined as

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}\, dt$$

Impulse is a measure of both strength and duration of collision force

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}\, dt = \int_{t_i}^{t_f} \frac{d\vec{p}}{dt}\, dt = \vec{p}_f - \vec{p}_i$$

Impulse momentum theorem for a particle

$$\vec{I}_{\rm net} = \Delta \vec{P}$$

Impulse momentum theorem for a system

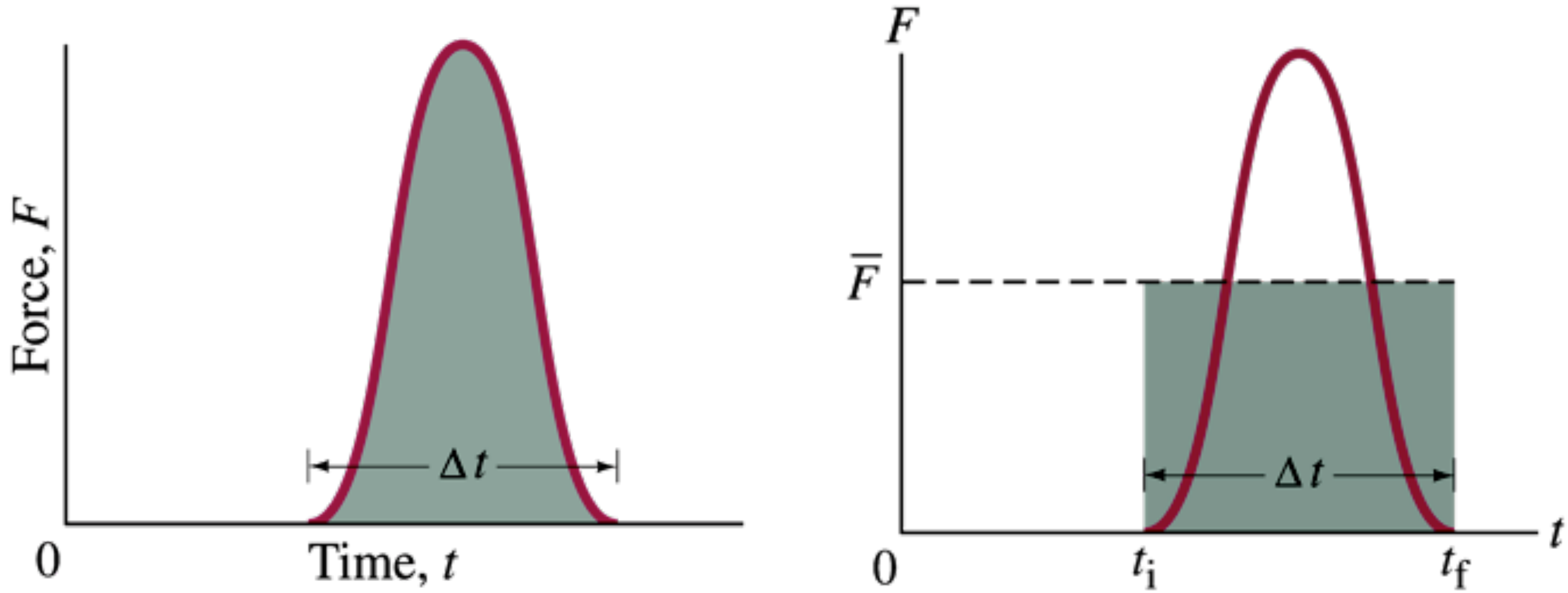
$$\vec{I}_{\rm net,\, ext} = \int_{t_i}^{t_f} \vec{F}_{\rm net,\, ext}\, dt = \Delta \vec{P}_{\rm sys}$$

Average force

Since time of collision is very short

we need not worry about exact time dependence of force

and can use average force



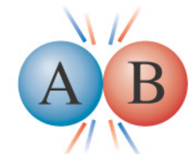
$$\bar{F} = \langle \vec{F} \rangle = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{F}\, dt = \frac{1}{\Delta t} \vec{I}$$

Conservation of Energy and Momentum in Collisions

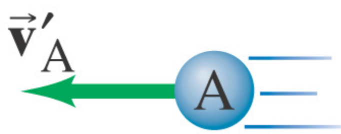
Momentum is conserved in all collisions

Collisions in which kinetic energy is conserved as well are called elastic collisions and those in which it is not are called inelastic

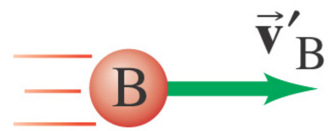
Approach



Collision



If elastic



If inelastic

Perfectly Elastic and Perfectly Inelastic Collisions

Conservation of momentum

$$m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}} = m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}}$$

In elastic collisions kinetic energy of system is conserved

$$\frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2 = \frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2$$

$$m_2 (v_{2\mathrm{f}}^2 - v_{2\mathrm{i}}^2) = m_1 (v_{1\mathrm{i}}^2 - v_{1\mathrm{f}}^2)$$

$$m_2 (v_{2\mathrm{f}} - v_{2\mathrm{i}})(v_{2\mathrm{f}} + v_{2\mathrm{i}}) = m_1 (v_{1\mathrm{i}} - v_{1\mathrm{f}})(v_{1\mathrm{i}} + v_{1\mathrm{f}})$$

From conservation of momentum

$$m_2 (v_{2\mathrm{f}} - v_{2\mathrm{i}}) = m_1 (v_{1\mathrm{i}} - v_{1\mathrm{f}})$$

Taking ratio of these two equations

$$(v_{2\mathrm{f}} + v_{2\mathrm{i}}) = (v_{1\mathrm{f}} + v_{1\mathrm{i}})$$

Rearranging we obtain relative velocities in an elastic collision

$$(v_{1\mathrm{i}} - v_{2\mathrm{i}}) = (v_{2\mathrm{f}} - v_{1\mathrm{f}})$$

In perfectly inelastic collisions objects have same velocity after collision (often because they stuck together)

$$v_{1\mathrm{f}} = v_{2\mathrm{f}} = v_{\mathrm{cm}}$$

A meteor whose mass was about 10^8 kg struck Earth with a speed of about 15 km/s and came to rest in Earth

ⓐ What was Earth's recoil speed?

ⓑ What fraction of meteor's kinetic energy was transformed to kinetic energy of Earth?

ⓒ By how much did Earth kinetic energy change as a result of this collision?

$$M_{\oplus} = 6 \times 10^{24} \text{ kg}$$

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(a) $$m_{\rm meteor} v_{\rm meteor} = (m_{\rm meteor} + M_\oplus) v_f$$

$$v_f = 6 \times 10^{-11}\ {\rm m/s}$$

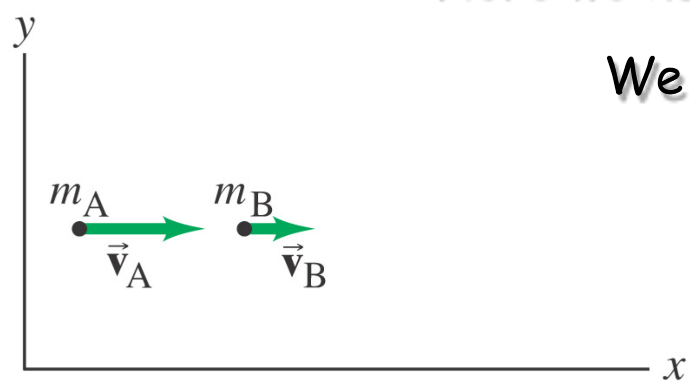
(b) $$\frac{K_f^{\rm Earth}}{K_i^{\rm meteor}} = 8 \times 10^{-15}$$

(c) $$\Delta K_{\rm Earth} = K_f^{\rm Earth} - K_i^{\rm Earth} = \frac{1}{2} M_\oplus v_f^2 = 10,800\ {\rm J}$$

Elastic Collisions in One Dimension

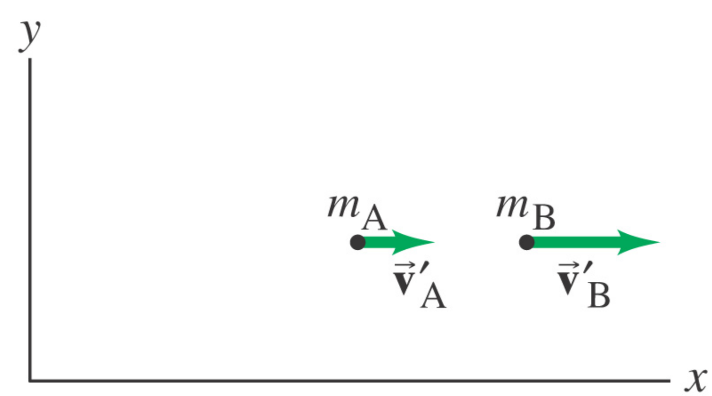
Here we have two objects colliding elastically

We know masses and initial speeds



Since both momentum and kinetic energy are conserved we can write two equations

This allows us to solve for two unknown final speeds

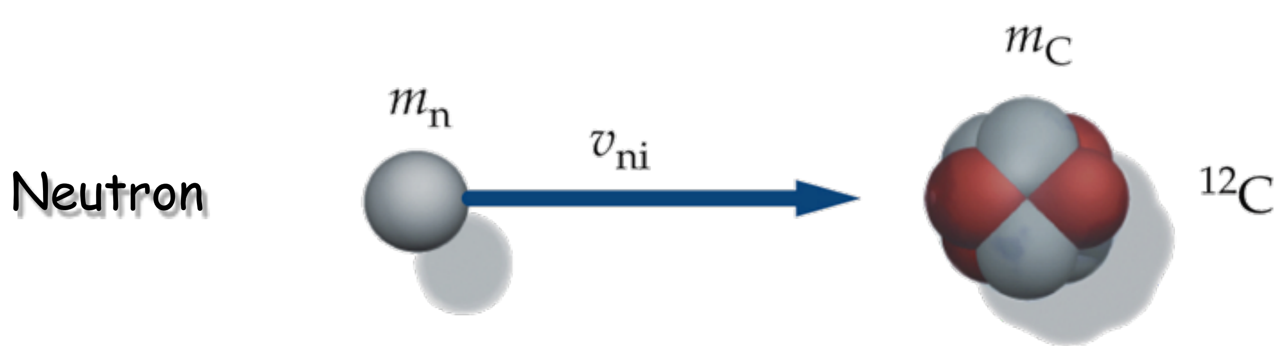


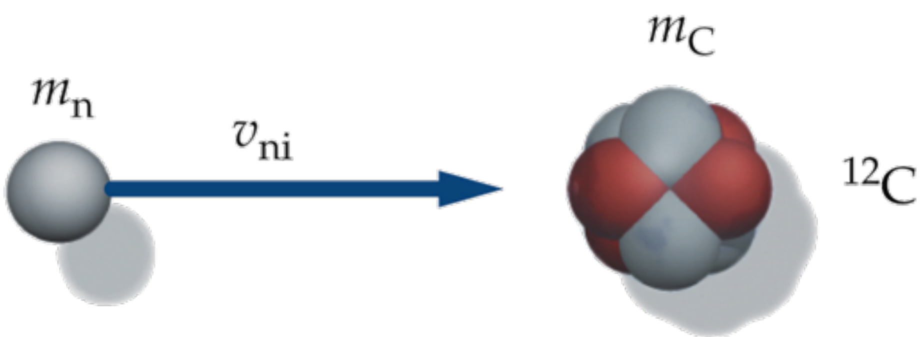
Elastic collision of a neutron and a nucleus

A neutron of mass $m_{\rm n}$ and speed $v_{\rm ni}$ undergoes a head-on elastic collision with a carbon nucleus of mass m_C initially atrest

(a) What are final velocities of both particles?

(b)What fraction f of its initial kinetic energy does neutron lose?





$$m_n v_{n_i} = m_n v_{n_f} + m_C v_{C_f}$$

for elastic collision $v_{C_f} - v_{n_f} = v_{n_i} \Rightarrow v_{C_f} = v_{n_i} + v_{n_f}$

$$m_n v_{n_i} = m_n v_{n_f} + m_C (v_{n_i} + v_{n_f})$$

$$v_{n_f} = -\frac{m_C - m_n}{m_n + m_C} v_{n_i}$$

$$v_{C_f} = v_{n_i} - \frac{m_C - m_n}{m_n + m_C} v_{n_i} = \frac{2m_n}{m_n + m_C} v_{n_i}$$

$$f = -\frac{\Delta K_n}{K_{n_i}} = \frac{K_{C_f}}{K_{n_i}} = \frac{m_C}{m_n} \left(\frac{v_{C_f}}{v_{n_i}} \right)^2 = \frac{4 m_n m_C}{(m_n + m_C)^2}$$

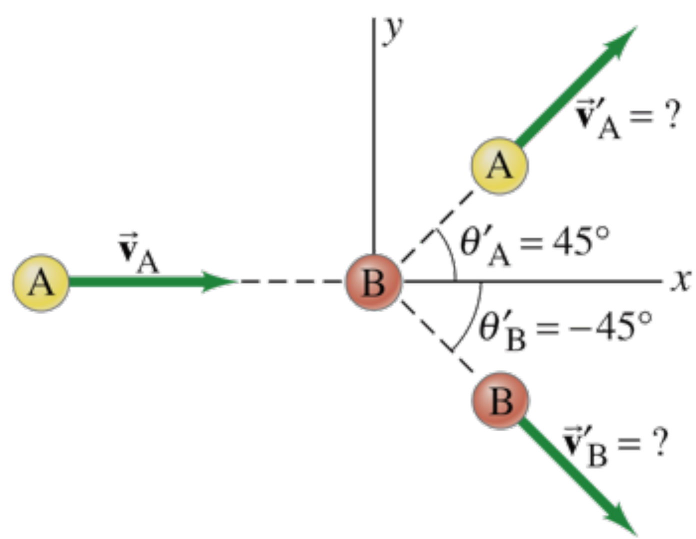
Collisions in Two or Three Dimensions

Conservation of energy and momentum can also be used to analyze collisions in two or three dimensions but unless situation is very simple math quickly becomes unwieldy

If moving object collides with an object initially at rest

knowing masses and initial velocities is not enough

we need to know angles as well in order to find final velocities

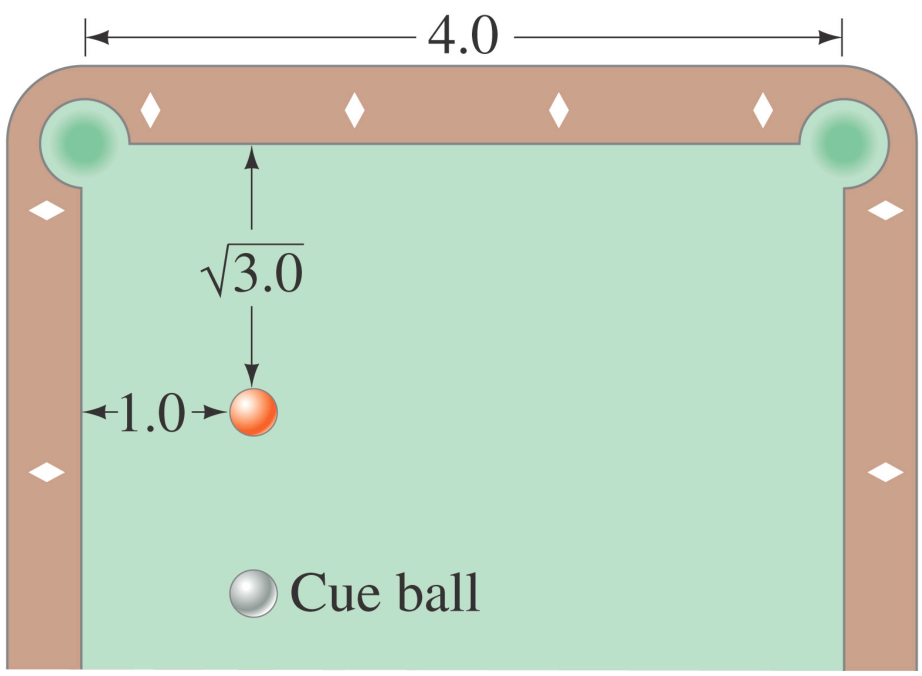


A novice pool player is faced with corner pocket shot shown in figure

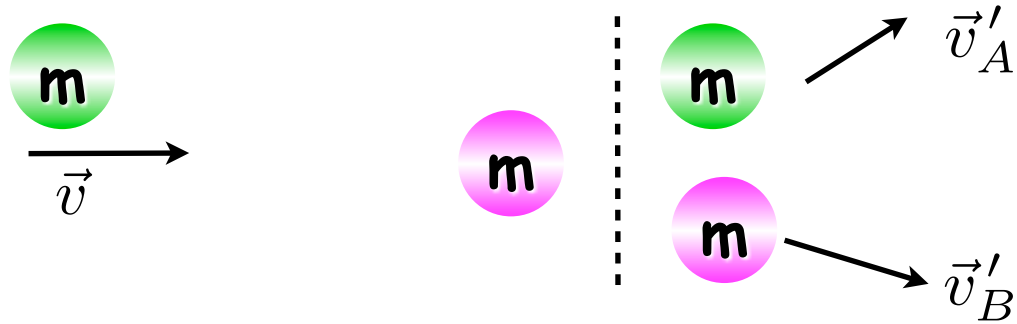
Relative dimensions are also given

Should player be worried about this being a "scratch shot"

in which cue ball will also fall into a pocket?



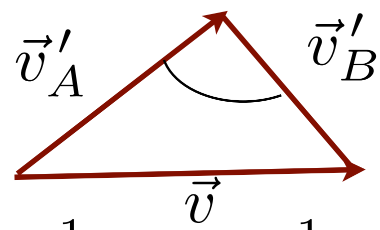
Give details



In elastic collision between two objects of equal mass with target at rest angle between final velocities of objects is 90°

Momentum conservation

$$m\vec{v} = m\vec{v}\,'_A + m\vec{v}\,'_B \Rightarrow \vec{v} = \vec{v}\,'_A + \vec{v}\,'_B$$



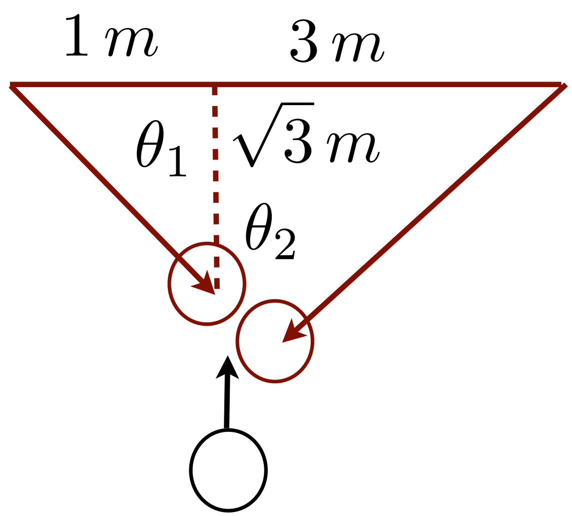
Kinetic energy conservation ☛ $\frac{1}{2}mv^2 = \frac{1}{2}m\,v'^2_A + \frac{1}{2}m\,v'^2_B \Rightarrow v^2 = v'^2_A + v'^2_B$

Applying law of cosines ☛ $v^2 = \vec{v}\,'^2_A + \vec{v}\,'^2_B - 2\,\vec{v}\,'_A\,\vec{v}\,'_B \cos\theta$

Equating two expressions for v^2 leads to

$$v'^2_A + v'^2_B = v'^2_A + v'^2_B - 2\,v'_A\,v'_B \cos\theta$$

$$\cos\theta = 0 \Rightarrow \theta = 90^\circ$$



Assume that target ball is hit correctly so that it goes in pocket

From geometry of right triangle

$$\theta_1 = \arctan[1/\sqrt{3}] = 30^\circ$$

From geometry of left triangle

$$\theta_2 = \arctan[3/\sqrt{3}] = 60^\circ$$

Because balls will separate at 90° ☛ if target ball goes in pocket this does appear to be a good possibility of a scratch shot