Lesson 2

Physics 168

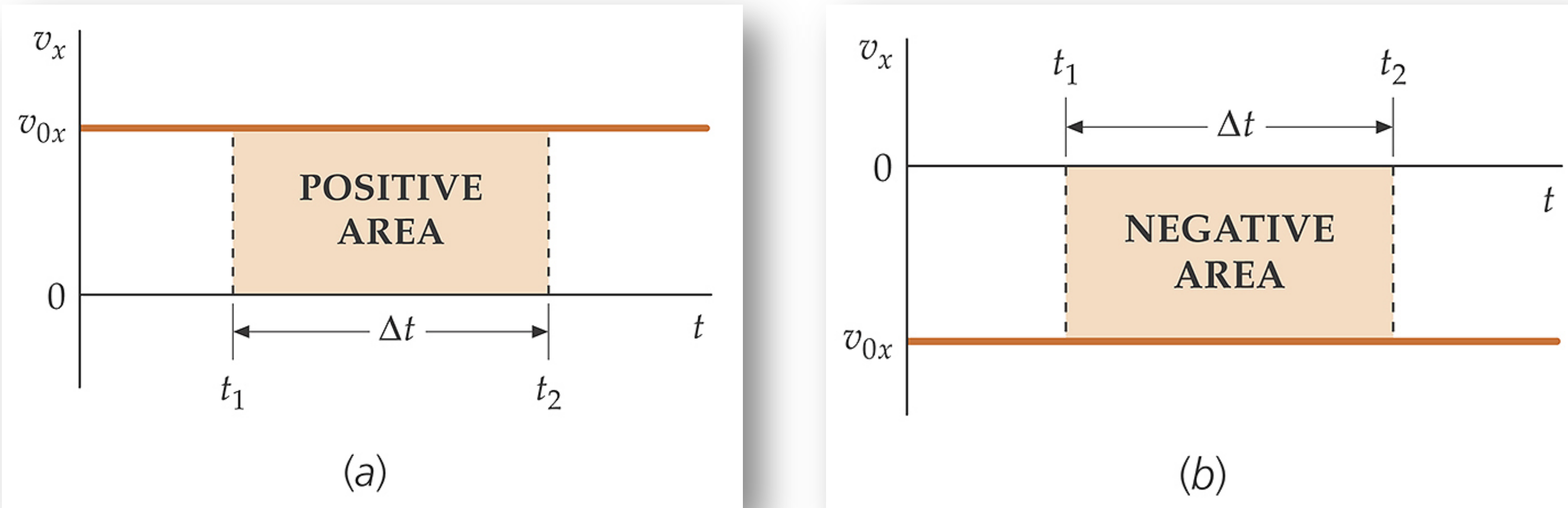
Luis Anchordoqui

Deriving Constant-Acceleration Kinematic Equations

To obtain an equation for position x as a function of time

look at special case of motion with constant velocity $v_x = v_0$

change in position Δx during an interval of time Δt is $\Delta x = v_0 \Delta t$



Area of shaded rectangle under v_x -versus- t

curve is height v_x times its width Δt

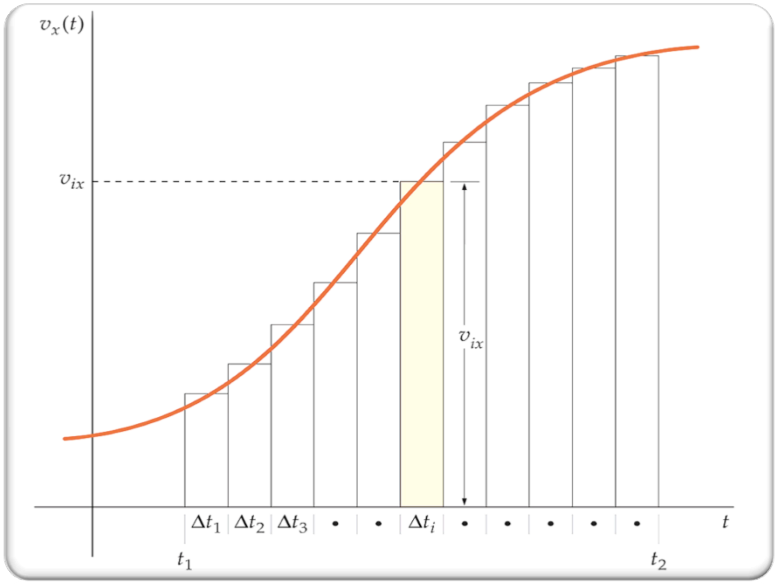
Area under curve is displacement Δx

Integral as Limit of Riemann Sum

Geometric interpretation of displacement

as area under v_x vs t curve

is true in general



To show this divide time interval

into numerous small intervals

Area of rectangle corresponding to ith interval Δt_i (shaded in figure)

$$v_{ix}\ \Delta t_i \sim \Delta x_i$$

Sum of rectangular areas $\approx$ sum of displacements during time interval

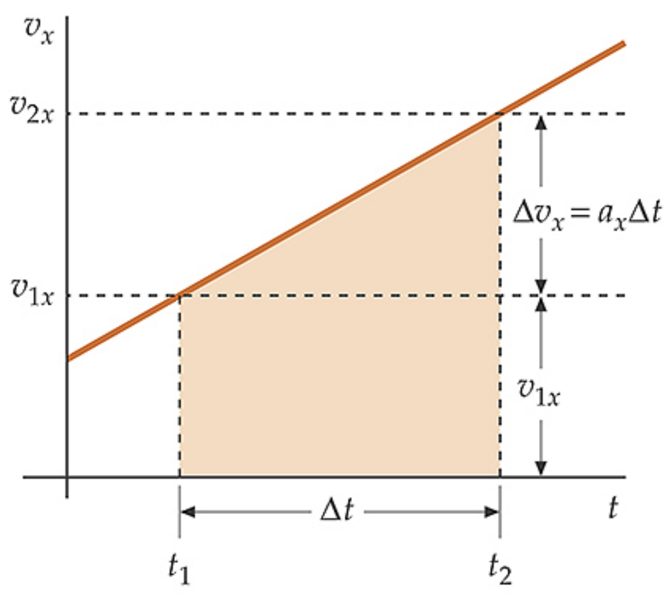
Limit of sum as Δt approaches to zero is called integral

$$\Delta x = x(t_2) - x(t_1) = \lim_{\Delta t \to 0} \left(\sum_i v_{ix}\ \Delta t_i \right) = \int_{t_1}^{t_2} v_x\ dt$$

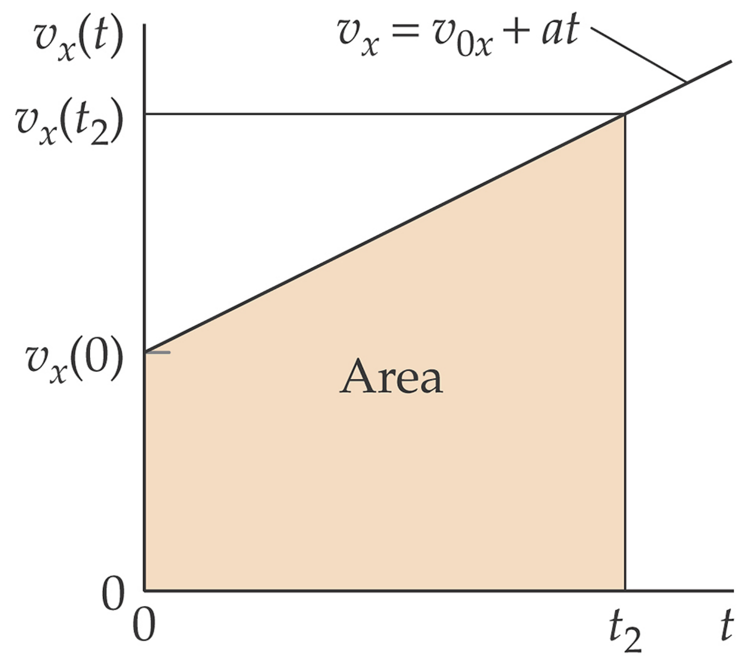
Example

For a motion with constant acceleration

Δx **is equal to area of shaded region** $\Delta x = v_x \Delta t + \frac{1}{2} a_x \Delta t^2$



(a)



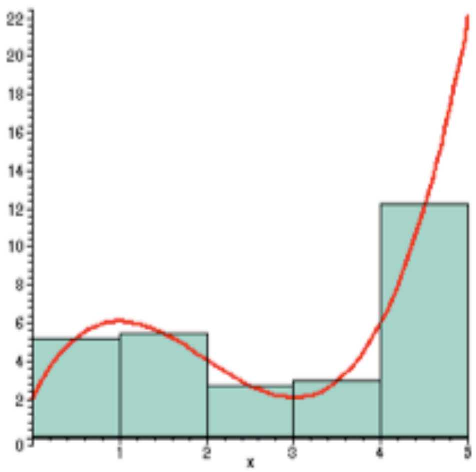
$$x(t_2) - x(t_1) = \int_0^{t_2} (v_{x_0} + a_x\ t)\ dt = v_{x_0} t + \frac{1}{2} a_x t^2 \Big|_0^{t_2} = v_{x_0} t_2 + \frac{1}{2} a_x t_2^2$$

Riemann Sum (Example)

$$f(x) = x^3 - 6x^2 + 9x + 2$$

$$\int_0^5 f(x)\,dx = 28.75$$

5 rectangles under the curve (rectangle width 1)



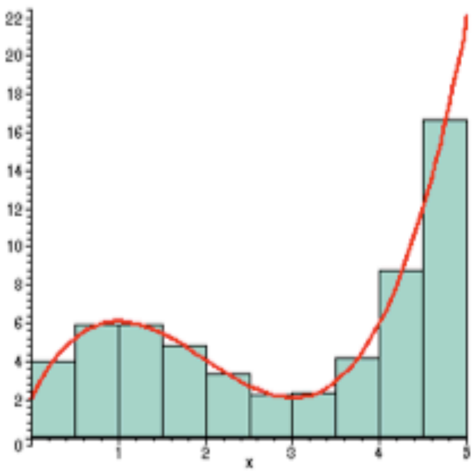
$$\int_0^5 f(x)\,dx \approx f(1/2) + f(3/2) + f(5/2) + f(7/2) + f(9/2)$$

$$= \sum_{i=0}^{4}\left[\left(i+\frac{1}{2}\right)^3 - 6\left(i+\frac{1}{2}\right)^2 + 9\left(i+\frac{1}{2}\right) + 2\right]$$

$$= 28.125$$

This is 2.17% less than the actual area

10 rectangles under the curve (rectangle width 1/2)



$$\int_0^5 f(x)\,dx \approx [f(1/4) + f(3/4) + \cdots + f(19/4)](1/2)$$

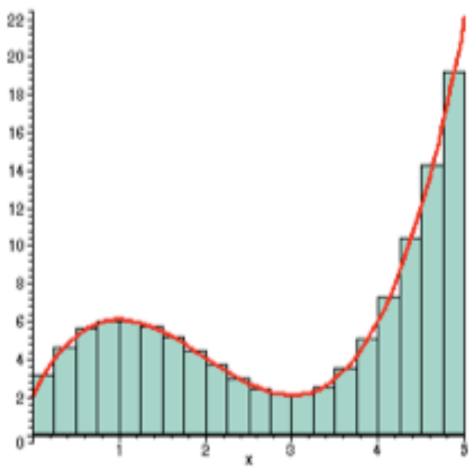
$$= \frac{1}{2}\left\{\sum_{i=0}^{9}\left[\left(\frac{i}{2}+\frac{1}{4}\right)^3 - 6\left(\frac{i}{2}+\frac{1}{4}\right)^2 + 9\left(\frac{i}{2}+\frac{1}{4}\right) + 2\right]\right\}$$

$$= 28.59375$$

This is 0.543% less than the actual area

Riemann Sum (Example)

20 rectangles under the curve (rectangle width 1/4)



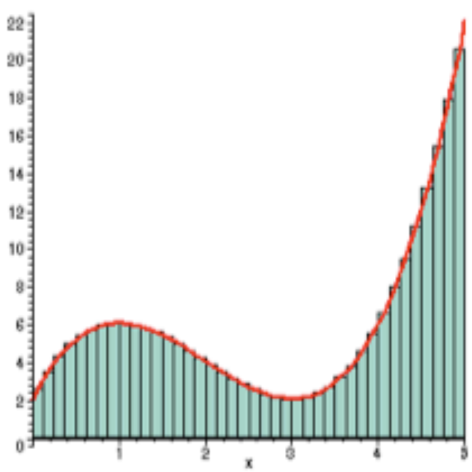
$$\int_0^5 f(x)\,dx \approx [f(1/8) + f(3/8) + \cdots + f(39/8)](1/4)$$

$$= \frac{1}{4}\left\{\sum_{i=0}^{19}\left[\left(\frac{i}{4}+\frac{1}{8}\right)^3 - 6\left(\frac{i}{4}+\frac{1}{8}\right)^2 + 9\left(\frac{i}{4}+\frac{1}{8}\right)+2\right]\right\}$$

$$= 28.7109375$$

This is 0.135% less than the actual area

40 rectangles under the curve (rectangle width 1/8)



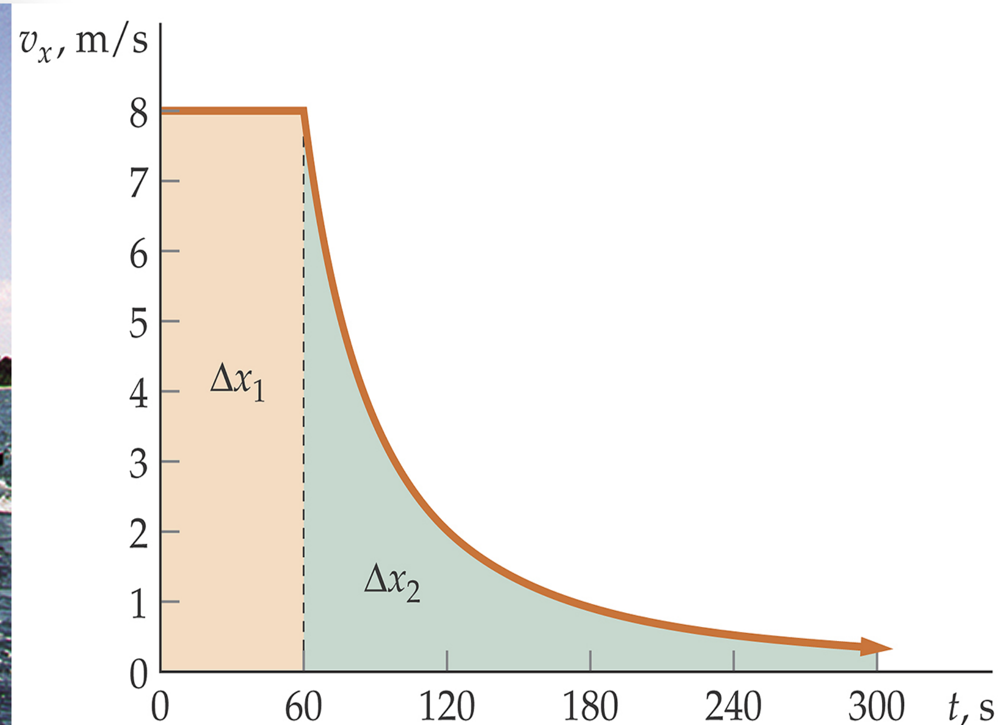
$$\int_0^5 f(x)\,dx \approx [f(1/16) + f(3/16) + \cdots + f(79/16)](1/8)$$

$$= \frac{1}{8}\left\{\sum_{i=0}^{39}\left[\left(\frac{i}{8}+\frac{1}{16}\right)^3 - 6\left(\frac{i}{8}+\frac{1}{16}\right)^2 + 9\left(\frac{i}{8}+\frac{1}{16}\right)+2\right]\right\}$$

$$= 28.74023438$$

This is 0.034% less than the actual area

A coasting boat



A Shelter Island ferryboat moves with constant velocity $v_0 = 8$ m/s for $T = 60$ s. It then shuts off its engines and coasts.

Its coasting velocity is given by $v_x = v_0 T^2/t^2$

What is displacement of the boat for interval $0 < t <$ infinity?

$$\Delta x_1 = v_{0_x} T = 8 \text{ m/s} \times 60 \text{ s} = 480 \text{ m}$$

$$\begin{aligned}
\Delta x_2 &= \int_T^\infty v_x \; dt \\
&= \int_T^\infty \frac{v_{0_x} T^2}{t^2} dt \\
&= v_{0_x} T^2 \int_T^\infty t^{-2} \; dt \\
&= v_{0_x} T^2 \left(-\frac{1}{t} \right) \Bigg|_T^\infty \\
&= v_{0_x} T = 480 \text{ m}
\end{aligned}$$

$$\Delta x = \Delta x_1 + \Delta x_2 = 960 \text{ m}$$

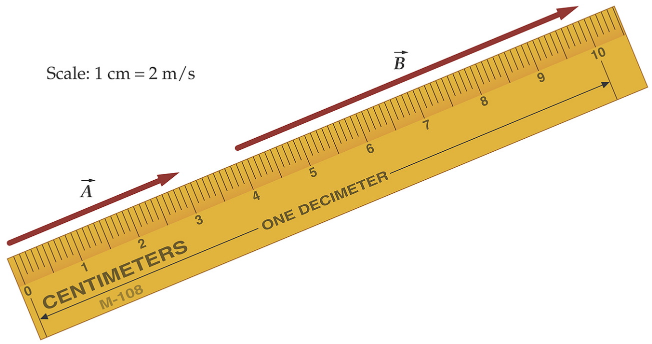

Motion in two and three dimensions
Luis Anchordoqui

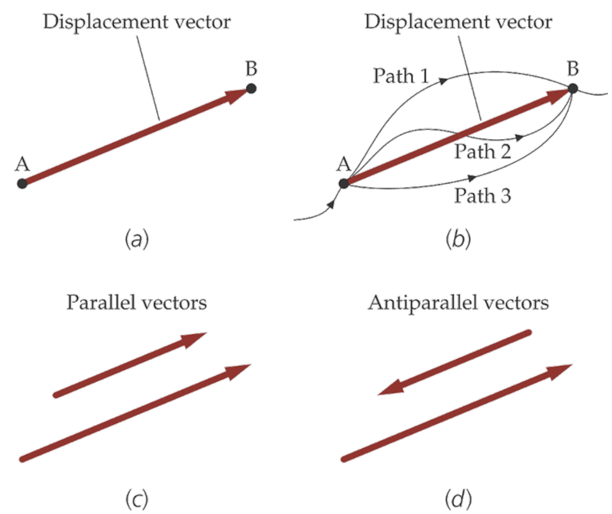
Vectors

Quantities that have magnitude and direction → vectors

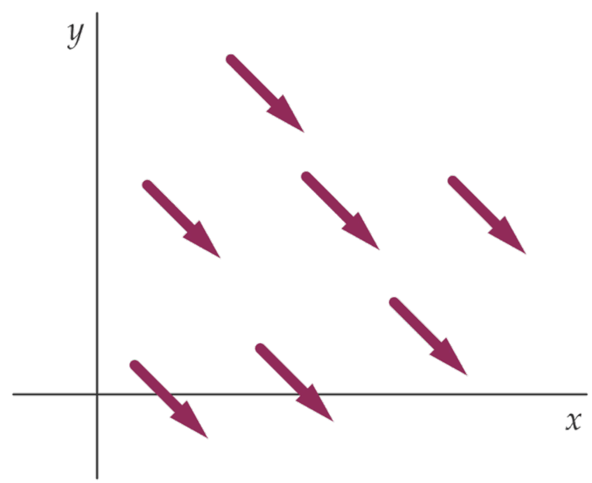
Quantities with magnitude but no associated direction → scalars

Examples

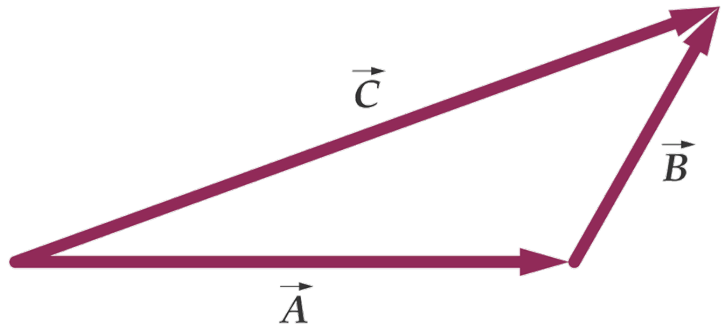




Vectors are equal if their magnitudes and directions are same

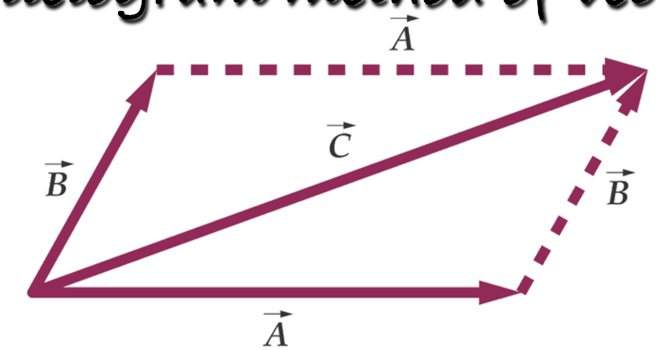


Addition of Vectors



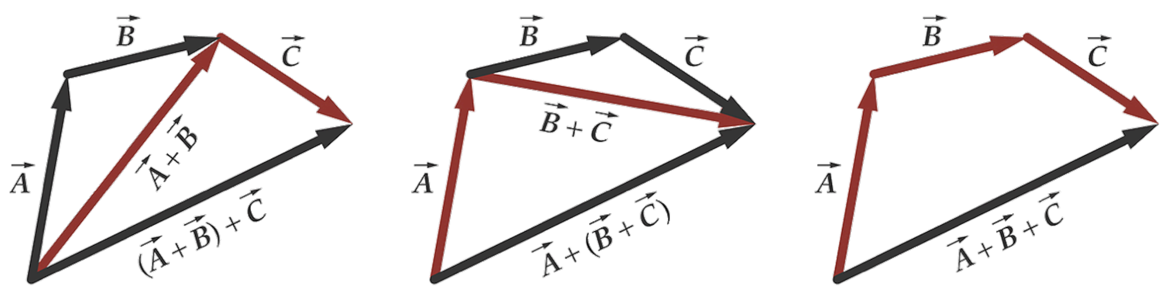
$$\vec{C} = \vec{A} + \vec{B}$$

Parallelogram method of vector addition

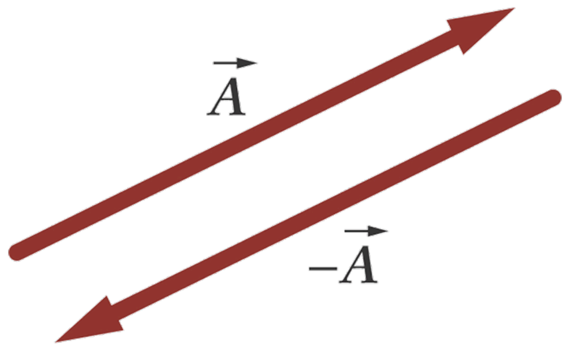


$$\vec{A} + \vec{B} = \vec{B} + \vec{A} = \vec{C}$$

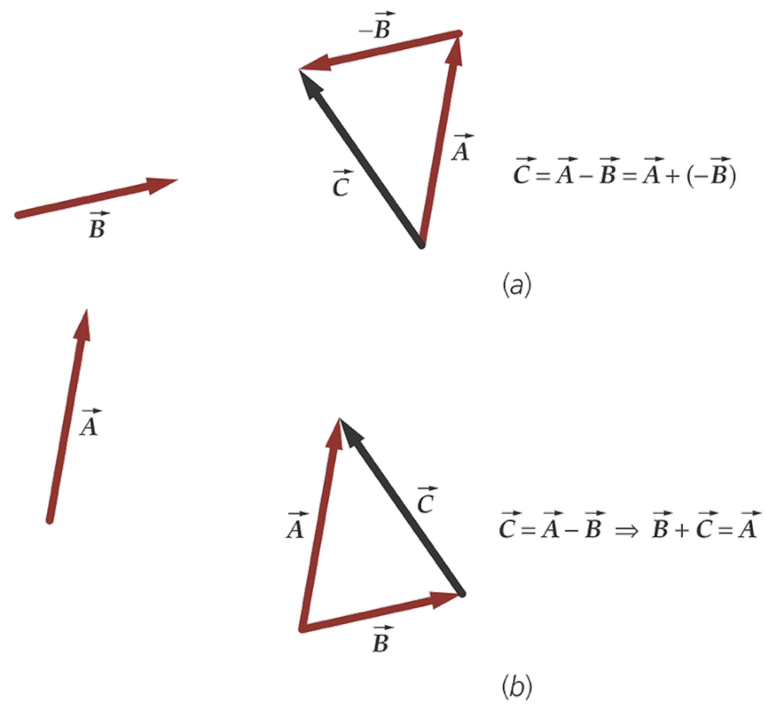
Vector addition is associative



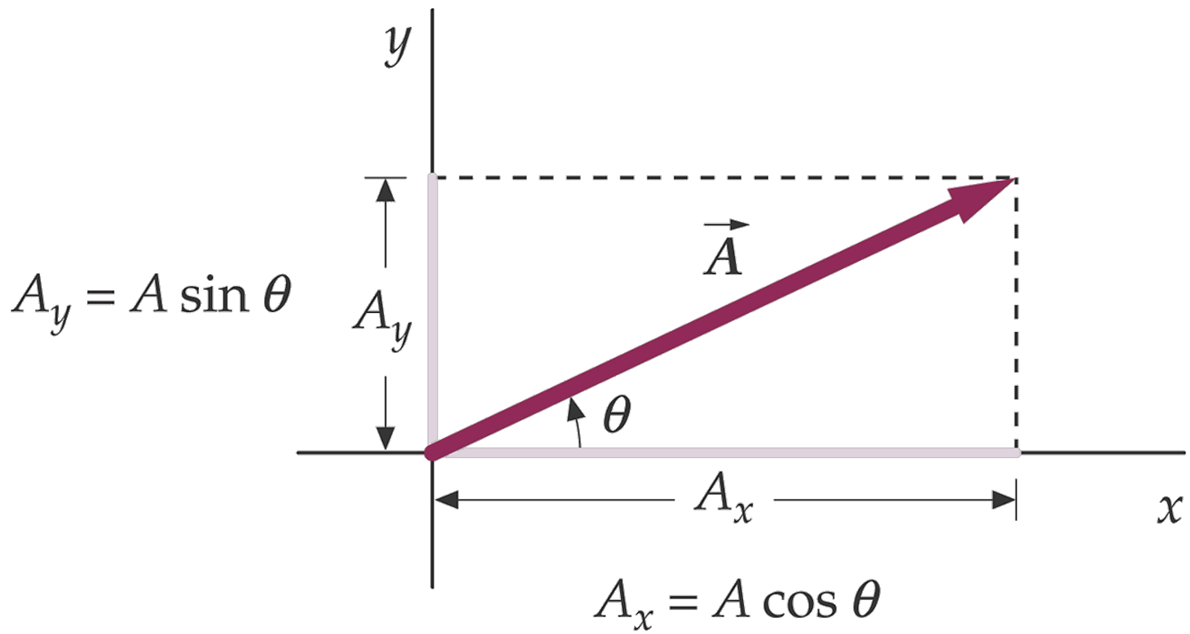
Subtraction of Vectors



$$\vec{A} - \vec{A} = \vec{A} + (-\vec{A}) = 0$$



Components of Vectors

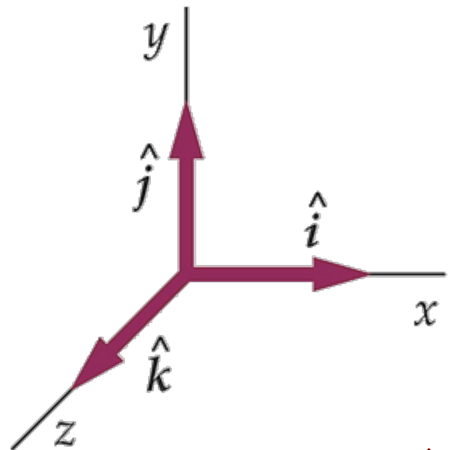


$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

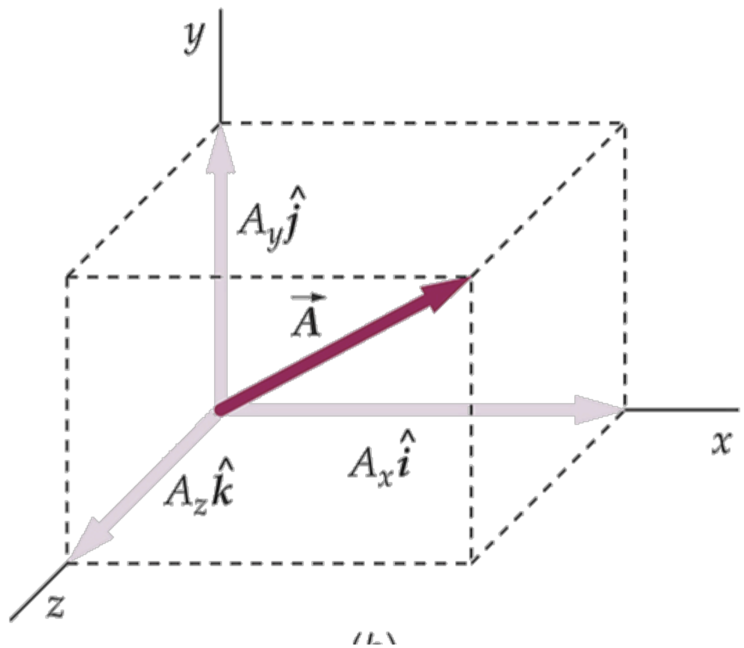
$$\tan\theta = \frac{A_y}{A_x}$$

Unit Vectors

A unit vector is a dimenionless vector with magnitude exactly equal to 1



Example $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

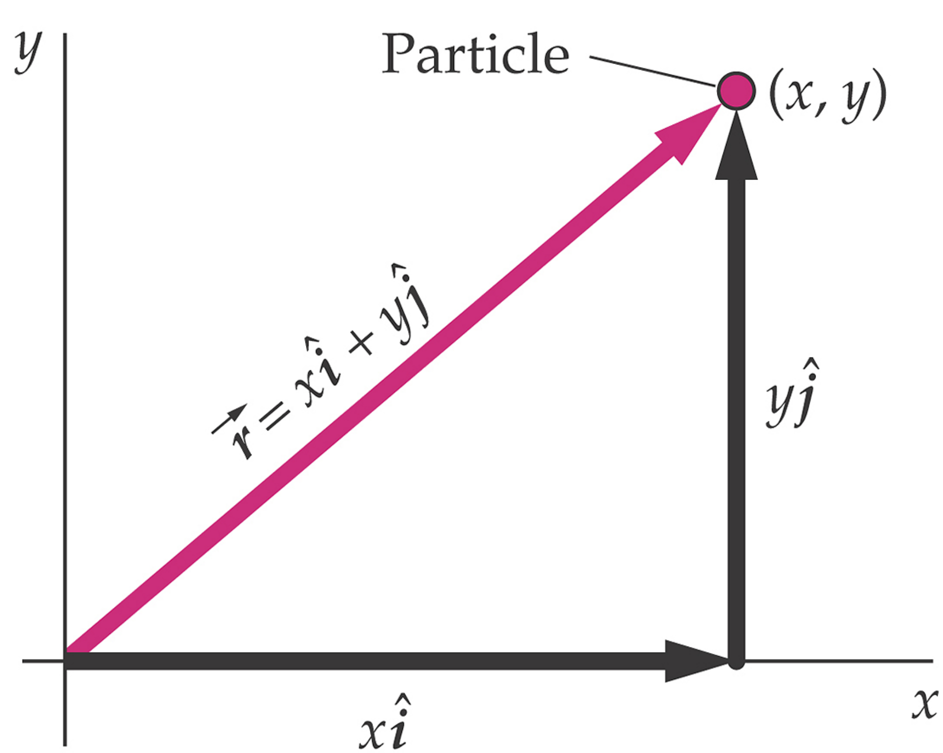


Position vector

Position vector of a particle is a vector drawn from origin of a coordinate system to location of a particle

For a particle in y-x plane at point with coordinates (x,y)

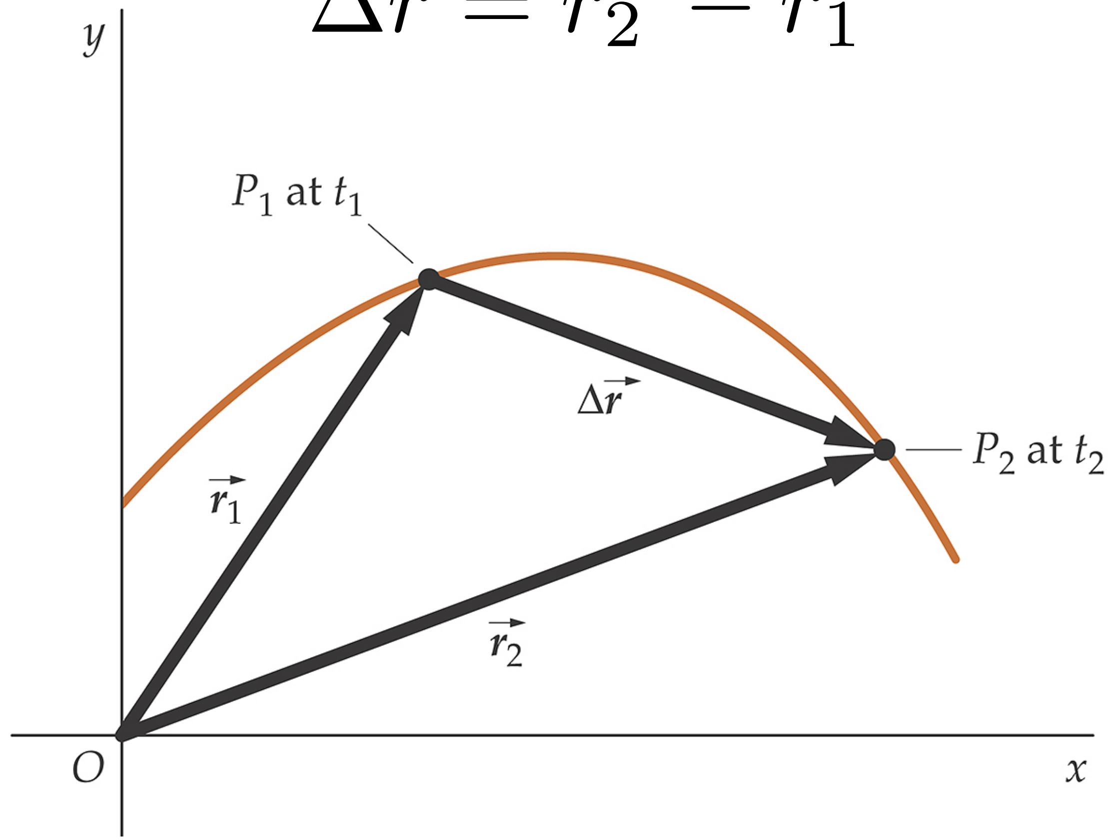
$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$



Displacement vector

Particle's change in position is displacement vector

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

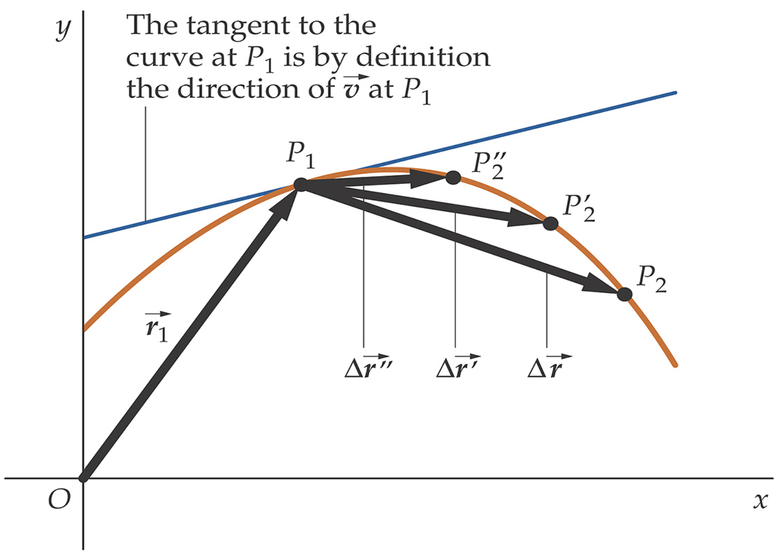


$$\Delta\vec{r} = (x_2 - x_1)\,\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} = \Delta x\,\hat{\imath} + \Delta y\,\hat{\imath}$$

Velocity vector

Average velocity vector $\vec{v}_{\text{av}} = \dfrac{\Delta\vec{r}}{\Delta t}$

Instantaneous velocity vector $\vec{v} = \lim_{\Delta t \to 0} \dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$



$$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x\hat{\imath} + v_y\hat{\jmath}$$

$$v = \sqrt{v_x^2 + v_y^2}$$

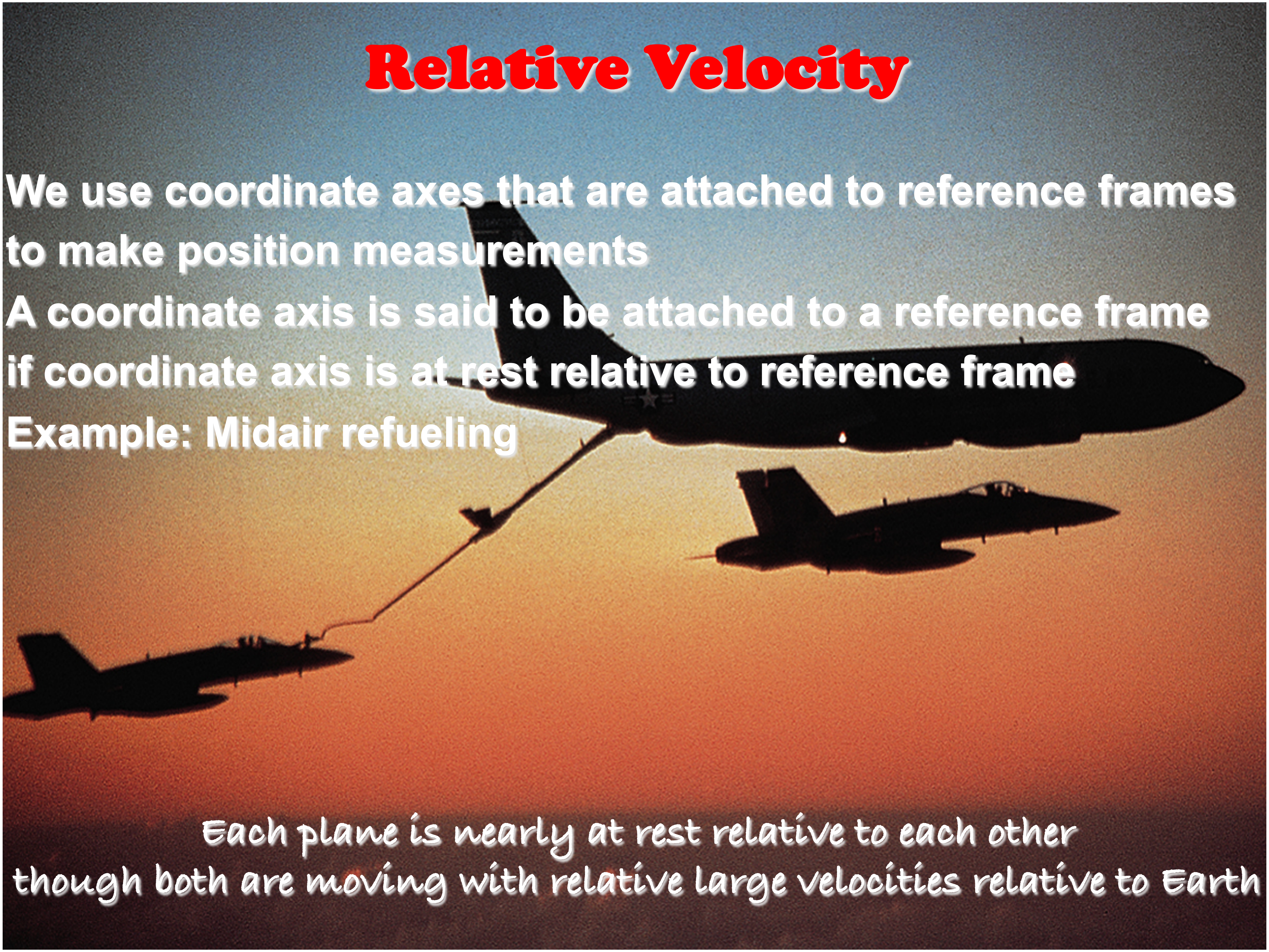
$$\theta = \arctan\left[\frac{v_y}{v_x}\right]$$

Relative Velocity

We use coordinate axes that are attached to reference frames to make position measurements

A coordinate axis is said to be attached to a reference frame if coordinate axis is at rest relative to reference frame

Example: Midair refueling



Each plane is nearly at rest relative to each other
though both are moving with relative large velocities relative to Earth

Relative Velocity (Cont'd)

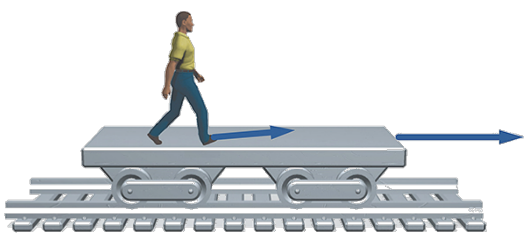
If particle p moves with velocity $\vec{v}_{\mathrm{pA}}$ relative to a reference frame A

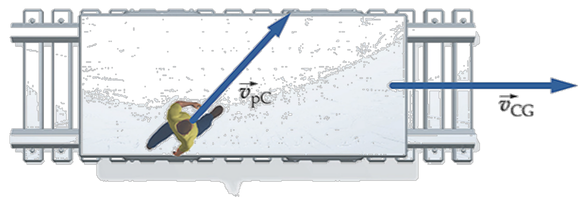
that is in turn moving with velocity $\vec{v}_{\mathrm{AB}}$ relative to a reference frame B

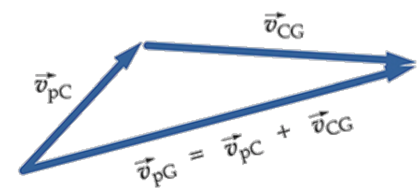
velocity $\vec{v}_{\mathrm{pB}}$ of particle relative to reference frame B is related to $\vec{v}_{\mathrm{pA}}$ & $\vec{v}_{\mathrm{AB}}$

by ☛ $\vec{v}_{\mathrm{pB}} = \vec{v}_{\mathrm{PA}} + \vec{v}_{\mathrm{AB}}$

Example







If a person p is on a railroad car C

that is moving with velocity $\vec{v}_{\mathrm{CG}}$ relative to ground G

and person is walking with velocity $\vec{v}_{\mathrm{pC}}$ relative to car

then velocity of person relative to G is vector sum of these two velocities

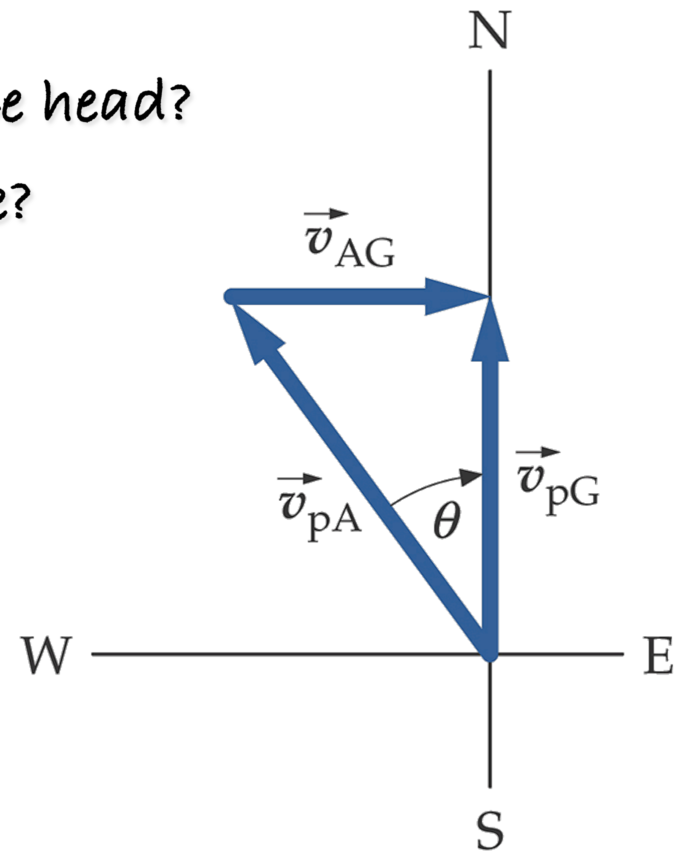
$$\vec{v}_{\mathrm{pG}} = \vec{v}_{\mathrm{pC}} + \vec{v}_{\mathrm{CG}}$$

A Flying Plane

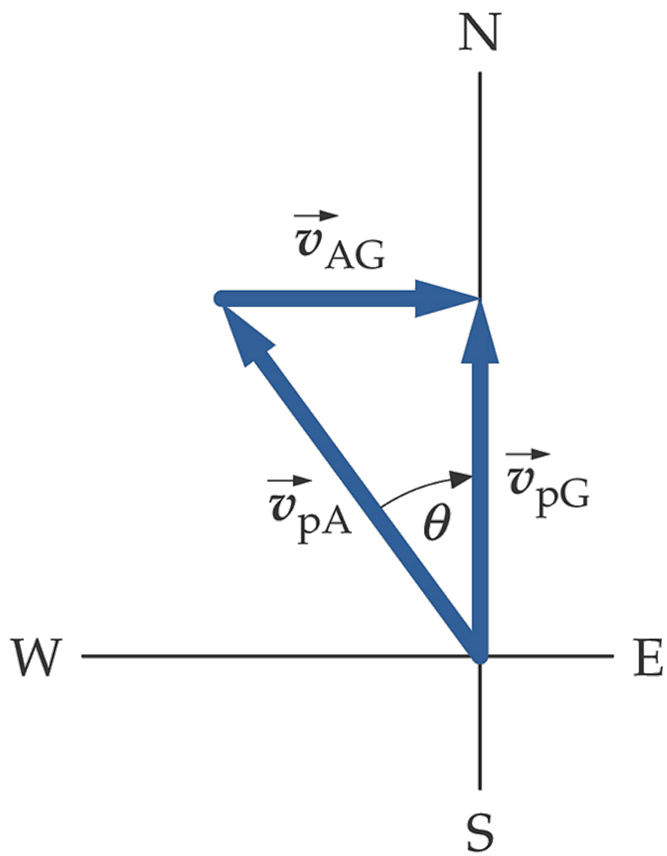
A pilot wishes to fly a plane due north relative to ground Airspeed of plane is 200 km/h and wind is blowing from west to east at 90 km/h.

(a) In which direction should plane head?

(b) What is ground speed of plane?



A Flying Plane



$$\sin\theta = v_{\mathrm{AG}}/v_{\mathrm{pA}} \Rightarrow \theta = 27^\circ \text{ west of north}$$

$$v_{pG} = \sqrt{v_{pA}^2 - v_{AG}^2} = 179 \text{ km/h}$$

Acceleration vector

Average acceleration vector

$$\vec{a}_{\mathrm{av}} = \frac{\Delta \vec{v}}{\Delta t}$$

Instantaneous acceleration vector

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k} = \frac{dx}{dt} \hat{\imath} + \frac{dy}{dt} \hat{\jmath} + \frac{dz}{dt} \hat{k}$$

$$\vec{a} = a_x \hat{\imath} + a_y \hat{\jmath} + a_z \hat{k}$$

where

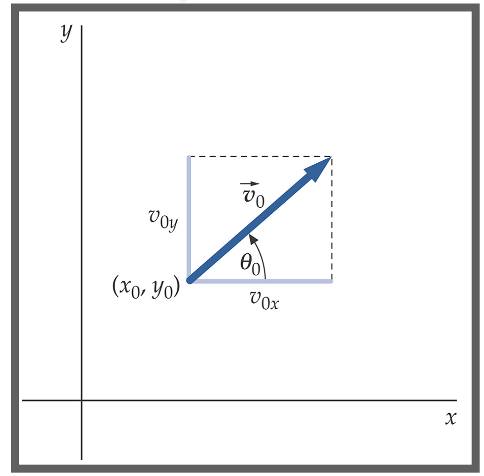
$$\vec{a} = \frac{dv_x}{dt} \hat{\imath} + \frac{dv_y}{dt} \hat{\jmath} + \frac{dv_z}{dt} \hat{k} = \frac{d^2 x}{dt^2} \hat{\imath} + \frac{d^2 y}{dt^2} \hat{\jmath} + \frac{d^2 z}{dt^2} \hat{k}$$

Projectile Motion

This type of motion occurs when on object is launched into air and is allowed to move freely. Initial velocity then has components

$$v_{0x} = v_0 \cos\theta_0$$

$$v_{0y} = v_0 \sin\theta_0$$



In absence of air resistance acceleration is constant

Component x of velocity is constant because no horizontal acceleration exists

$$a_x = 0$$

y component of velocity varies with time according to

$$a_y = -g$$

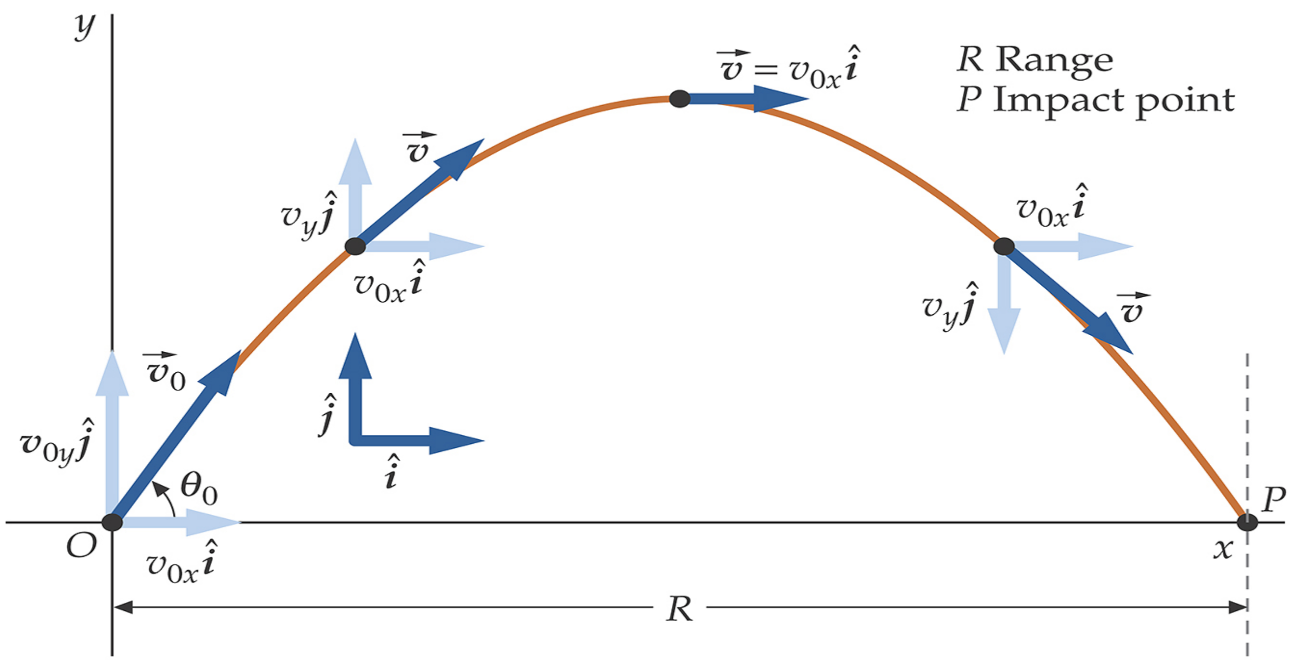
Path of a Projectile

Displacements x and y are given by

$$x(t) = x_0 + v_{0x}t \qquad y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Velocity components

$$v_x = v_{0x} \qquad v_y = v_{0y} - gt$$



Horizontal Range of a Projectile

Horizontal range of a projectile can be written in terms of its initial speed and initial angle above horizontal

Flight time is obtained by setting $y = 0$

$$0 = v_{0y}t - \frac{1}{2}gt^2 \qquad t > 0$$

Flight time of projectile is thus

$$T = \frac{2v_{0y}}{g} = \frac{2v_0 \sin\theta_0}{g}$$

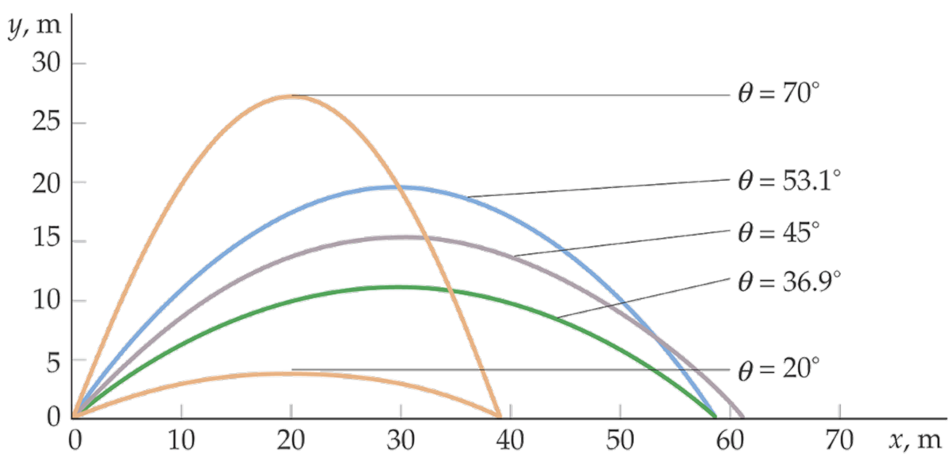
To find horizontal range we substitute flight time in x-equation of motion

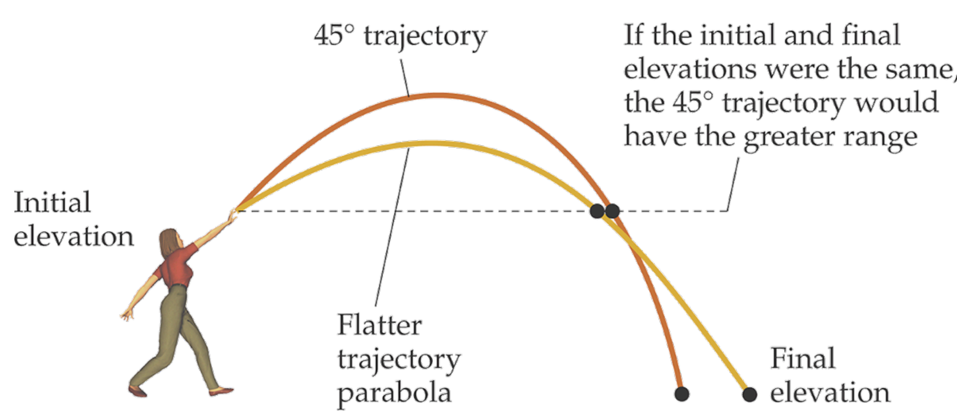
$$R = v_{0x}T = \frac{2v_0^2}{g} \sin\theta_0 \cos\theta_0$$

Horizontal Range of a Projectile

This can be further simplified by using trigonometric identity

$$\sin 2\vartheta = 2\sin\vartheta\ \cos\vartheta \Rightarrow R = \frac{v_0^2}{g}\ \sin(2\theta_0)$$





To catch a thief

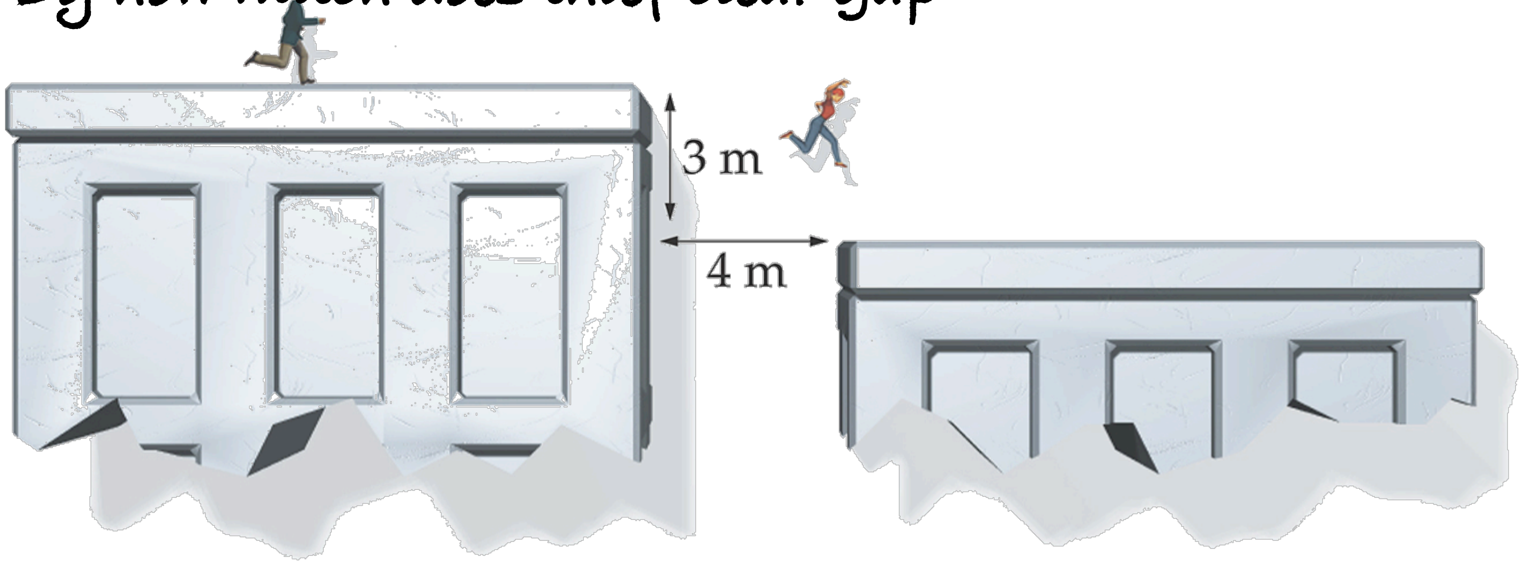
A police officer chases a master jewel thief across city rooftops They are both running when they come to a gap between buildings that is 4.00 m wide and has a drop of 3.00 m

Thief having studied a little of physics, leaps at 5.00 m/s at an angle of 45 degrees above horizontal, and clears gap easily

Police officer did not study physics and thinks he should maximize his horizontal velocity, so he leaps horizontally at 5.00 m/s.

(a) Does police clear gap

(b) By how much does thief clear gap



We write y(t) for the police officer and solve for t when y = 3 m

$$y = \frac{1}{2}gt^2 \Rightarrow t = 0.78 \text{ s}$$

By substituting this time in the x(t) equation we get

$$x = v_{0_x}t = 3.91 \text{ m}$$

Because 3.91 m < 4.00 m the police officer fails to make it across the buildings

We write y(t) for the thief and solve for t when y = 3 m & $v_{0_y} = -\frac{5\sqrt{2}}{2}$ m/s

$$y = v_{0_y}t + \frac{1}{2}gt^2 \Rightarrow t = 1.22 \text{ s}$$

Horizontal distance travelled by thief is

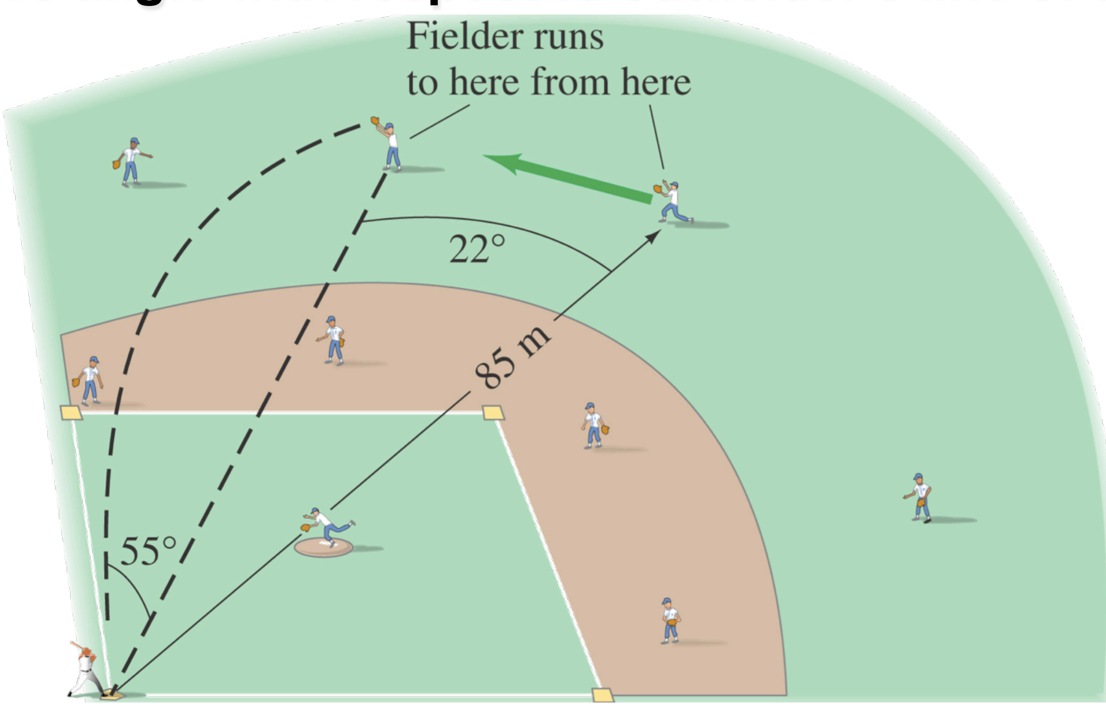
$$x = v_{0_x}t = 4.31 \text{ m}$$

$$\Delta x = 4.31 \text{ m} - 4.00 \text{ m} = 0.31 \text{ m}$$

At $t = 0$ a batter hits a baseball with an initial speed of $32\,\mathrm{m/s}$ at a 55° angle to horizontal. An outfielder is 85 m from batter at $t = 0$, and as seen from home plate, line of sight to outfielder makes a horizontal angle of 22° which plane in which ball moves

What speed and direction must fielder take in order to catch ball at same height from which it was struck?

Give angle with respect to outfielder's line of sight to home plate



The ball is being caught at the same hight from which it was struck

Set origin at position where ball was struck and so equations of motion are

$$\vec{r}(t) = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \qquad \vec{v}(t) = \vec{v}_0 + \vec{a}t$$

$$x(t) = v_{0_x} t \qquad y(t) = v_{0_y} t - \frac{1}{2} g t^2$$

$$v_{0_x} = v_0 \cos\theta_0 \qquad v_{0_y} = v_0 \sin\theta_0$$

We determine the time the ball was in the air by setting

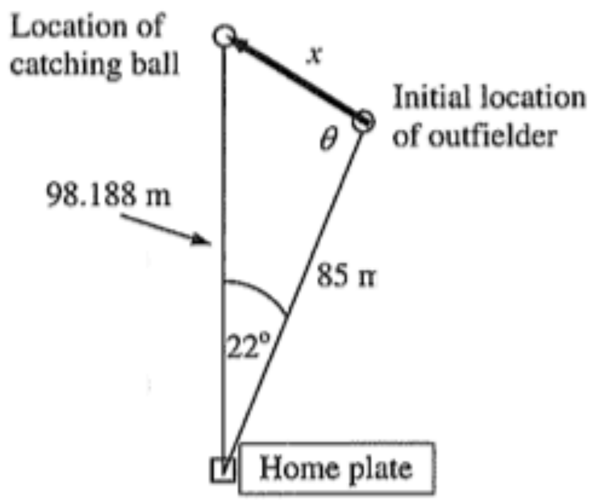
$$y(t) = 0 \Rightarrow t(v_{0_y} - tg/2) = 0 \quad \therefore \quad t_{\text{struck}} = 0 \ \wedge \ t_{\text{catch}} = 2v_{0_y}/g$$

Substitute time ball was in the air on x(t) equation to obtain catching position (a.k.a. ball range)

$$R = 2\frac{v_0^2}{g} \sin\theta_0 \ \cos\theta_0 = 98.19 \text{ m}$$

As seen from above:

location of home plate, point where ball must be caught, and outfielder initial location are:



The dark arrow shows the direction in which the outfielder must run

The length of the distance is found from the law of cosines as applied to the triangle

$$x = \sqrt{a^2 + b^2 - 2ab\cos\alpha}$$

$$= \sqrt{98.19^2 + 85^2 - 2 \times 98.19 \times 85 \times \cos 22^\circ} = 37.27 \text{ m}$$

The angle is found from the law of sines as applied to the triangle

$$\frac{\sin 22^\circ}{x} = \frac{\sin\theta}{98.19 \text{ m}} \Rightarrow \sin\theta = \left(\frac{98.19}{37.27}\sin 22^\circ\right) = 0.987 \;\therefore\; \theta = 80.7^\circ \;\wedge\; \theta = 99.1^\circ$$

Since $98.19^2 > 85^2 + 37.27^2$ angle must be obtuse and so we choose $\theta = 99.1^\circ$

Assume outfielder's time for running is same as time of ball flight

$$t_{\text{catch}} = 2v_{0_y}/g = 5.35 \text{ s}$$

Average velocity of outfielder must be

$$\langle v \rangle = \frac{\Delta d}{t} = \frac{37.27 \text{ m}}{5.5 \text{ s}} = 7 \text{ m/s}$$

@ angle of 99.1 degrees relative to outfielder's line of sight to home plate