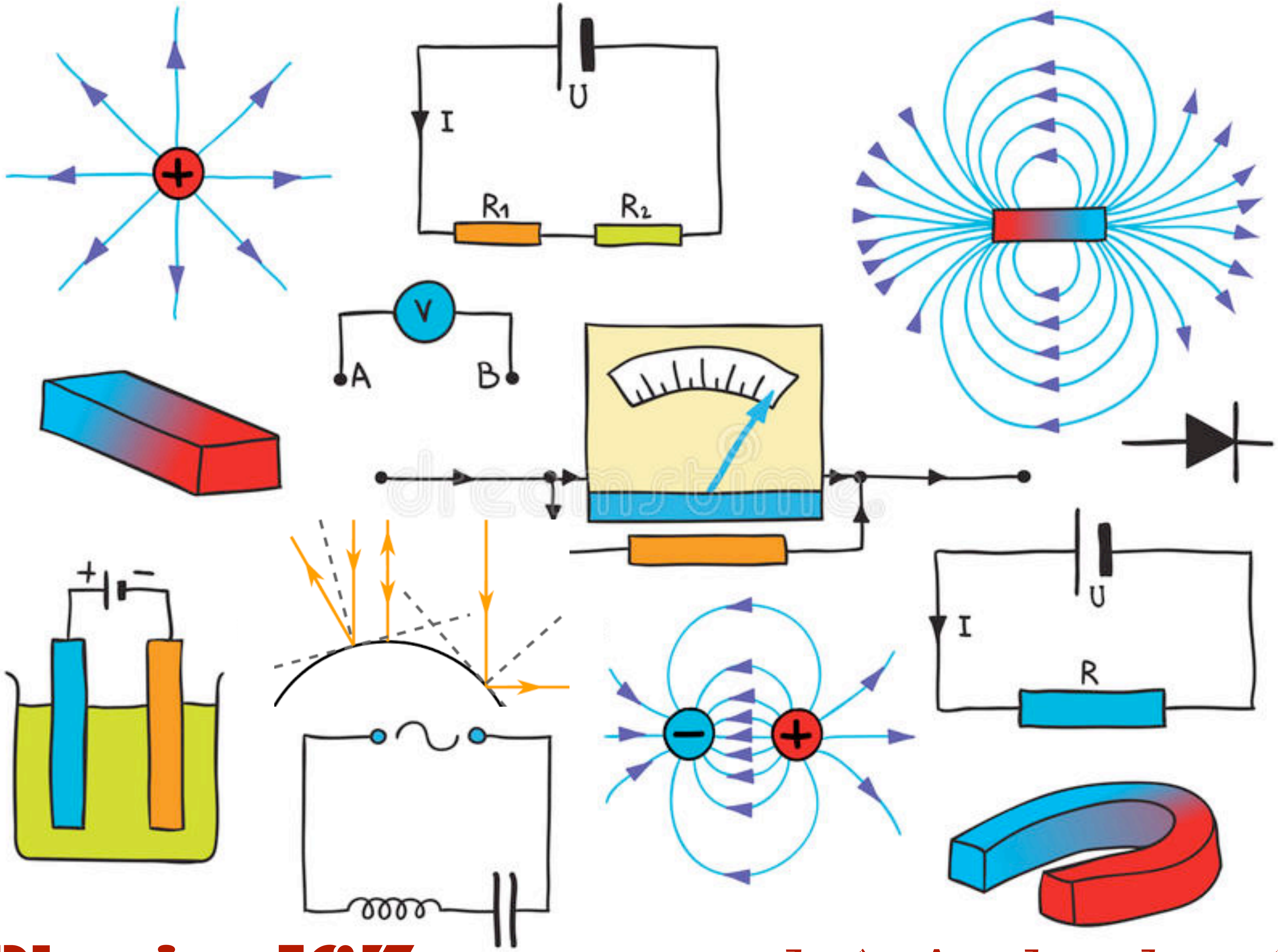


Physics 167

Luis Anchordoqui

From Coulomb's Law to Gauss's Law

➢ Try to calculate the electric field generated by

- ✱ a point charge **easy**
- ✱ an infinitely long straight wire with evenly distributed charge **hard**
- ✱ a wire loop **only at special locations**
- ✱ a round disk **only at special locations**
- ✱ an infinitely large plane **What??**
- ✱ a solid sphere with evenly distrubuted charge 

➢ Are there other ways to calculate electric field generated from a charge distribution?

➢ Electric field is generated by source charges ☛

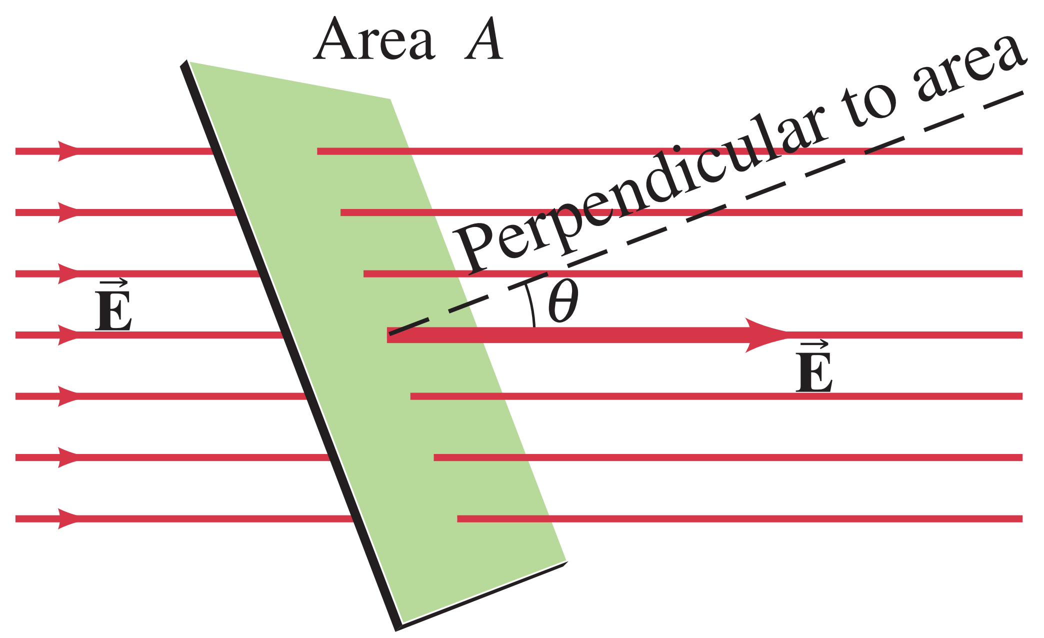
are there ways to connect electric field directly with this source charges?

The answer is YES

Electric flux

➢ Electric flux ☛ electric field passing through a given area

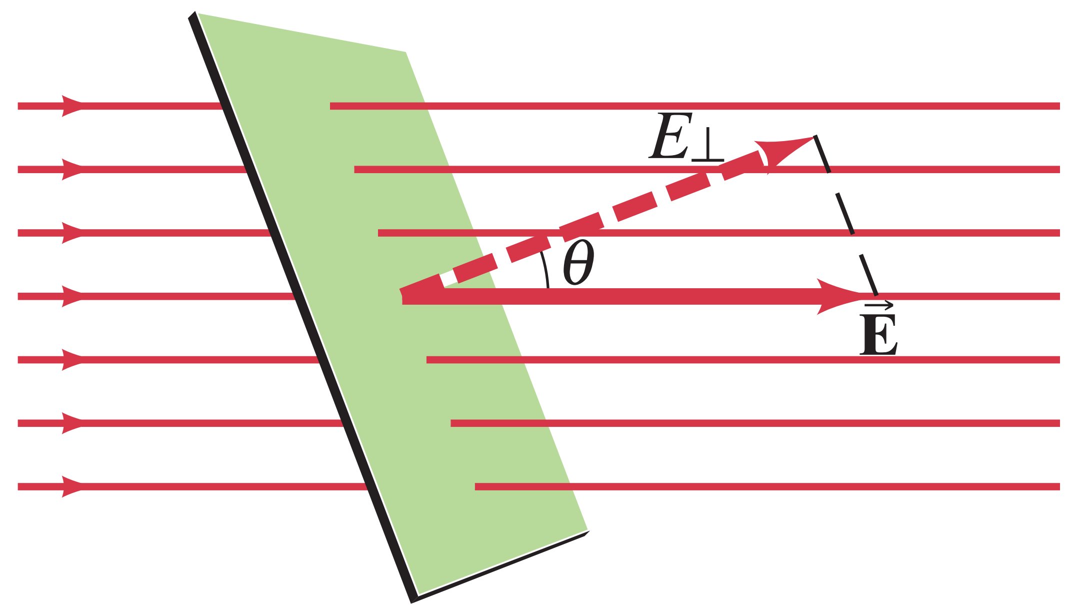
➢ For a uniform electric field E passing through area A ☛ electric flux is defined as



$$\Phi_E = EA \ \cos\theta$$

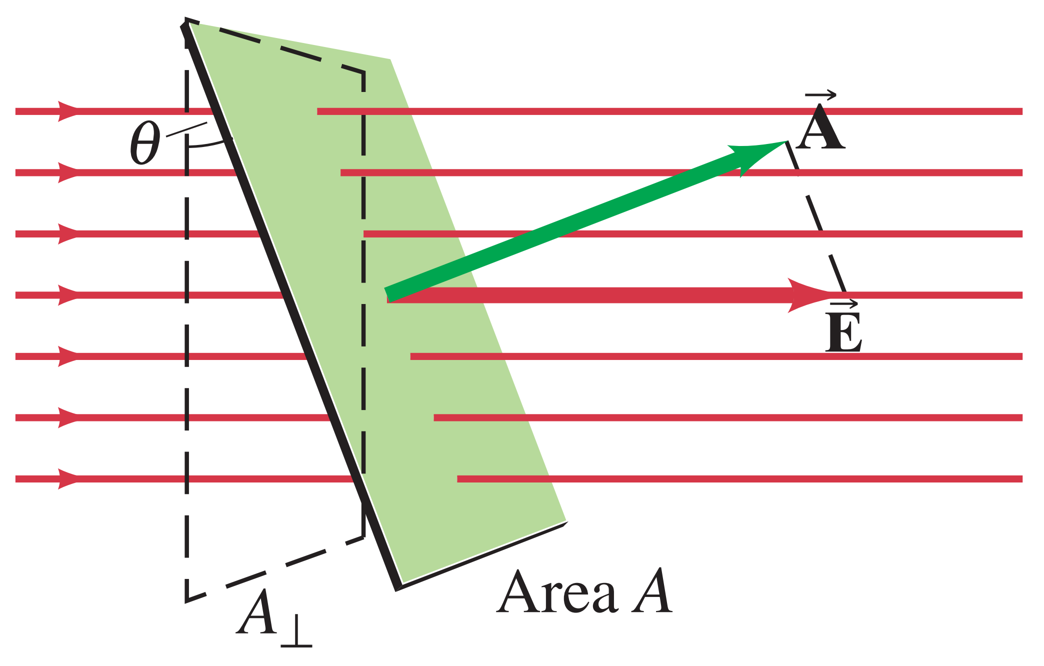
θ ☛ angle between the electric field direction and a line drawn perpendicular to area

➢ Flux can be written equivalently as



$$\Phi_E = E_\perp A = EA_\perp = EA \ \cos\theta$$

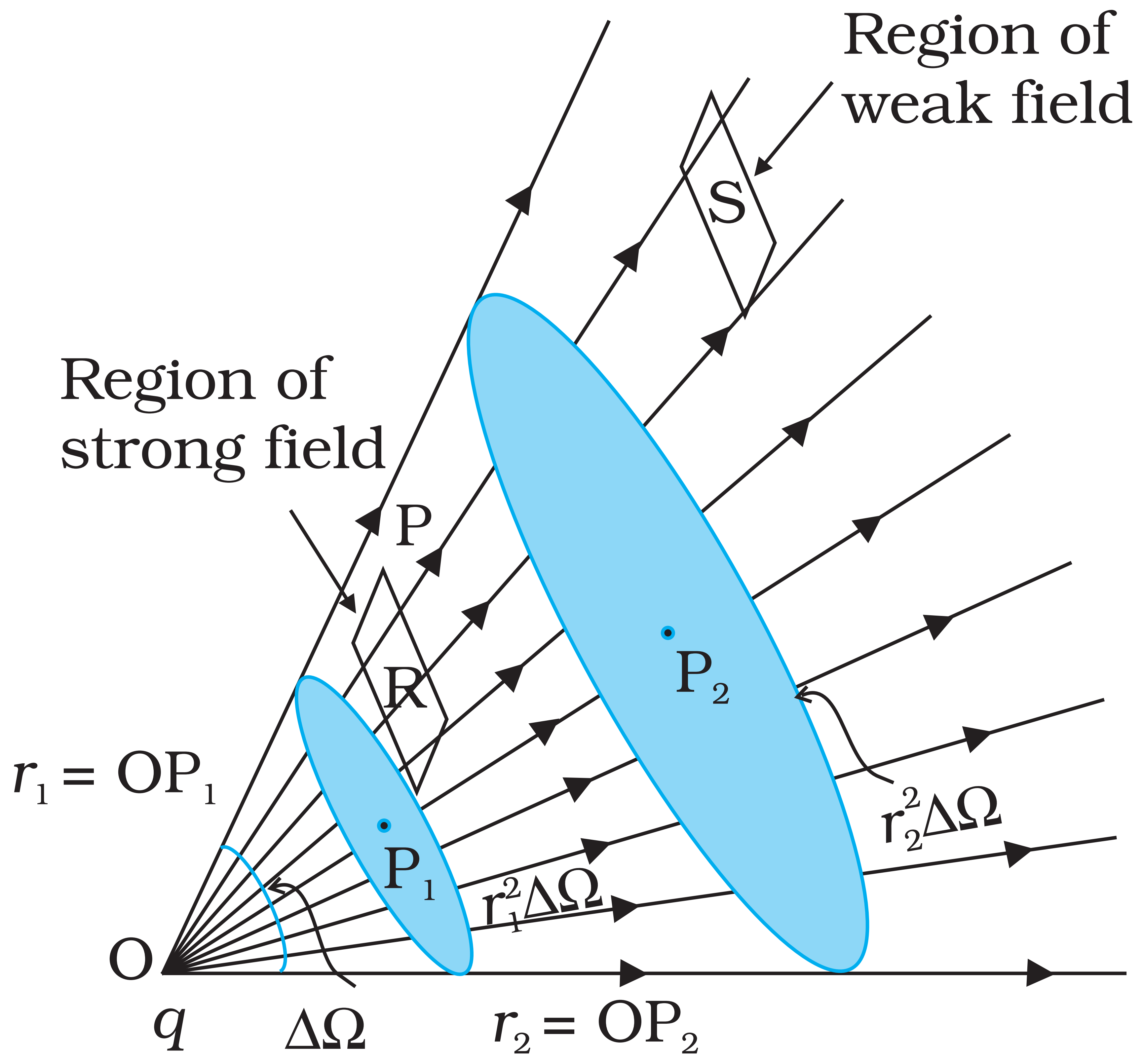
$E_\perp = E \ \cos\theta$ ☛ component of E perpendicular to area



$A_\perp = A \ \cos\theta$ ☛ projection of area A perpendicular to field E

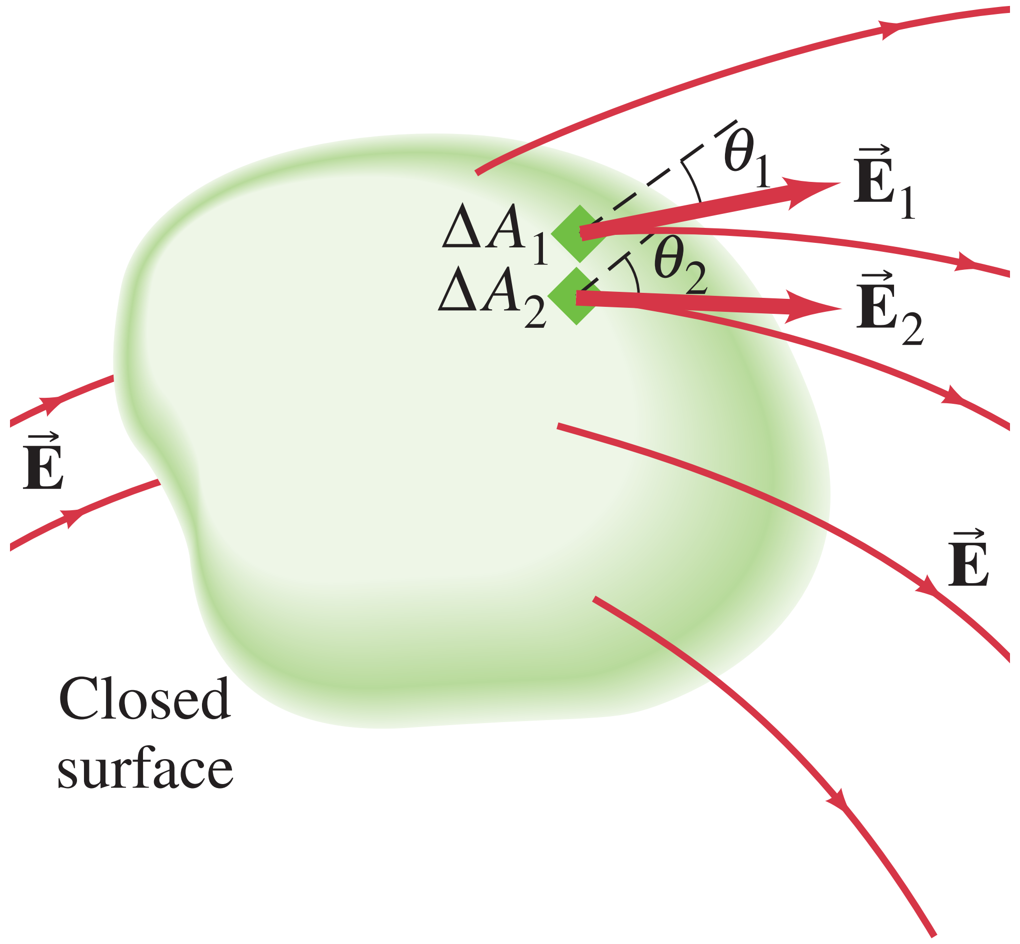
Electric flux

- Electric flux can be interpretated in terms of field lines
- Recall that field lines can always be drawn so that number (N) passing through unit area perpendicular to field ($A_\perp$) is proportional to magnitude of field (E)



Gauss Law

➢ Gauss's law involves the total flux through a closed surface ☛ a surface of any shape that encloses a volume of space



➢ For any such surface ☛ we divide the surface up into many tiny areas $\Delta A_1, \Delta A_2, \Delta A_3, \ldots$, and so on

➢ We make the division so that each ΔA is small enough that it can be considered flat and so that the electric field can be considered constant through each ΔA

➢ Then the total flux through the entire surface is the sum over all the individual fluxes through each of tiny areas

$$\Phi_E = E_1\ \Delta A_1\ \cos\theta_1 + E_2\ \Delta A_2\ \cos\theta_2 + \cdots + E_N\ \Delta A_N\ \cos\theta_N$$

$$= \sum_{j=1}^{N} E_j\ \Delta A_j\ \cos\theta_j = \sum_{j=1}^{N} E_{\perp j}\ \Delta A_j = \sum E_\perp\ \Delta A$$

Gauss Law

➢ Number of field lines starting on a positive charge or ending on a negative charge is proportional to magnitude of charge

➢ Hence ☛ the net number of lines N pointing out of any closed surface (number of lines pointing out minus the number pointing in) must be proportional to the net charge enclosed by the surface $Q_{\rm encl}$

➢ But the net number of lines N is proportional to the total flux ☛

$$\Phi_E = \sum_{\substack{\rm closed \\ \rm surface}} E_\perp \, \Delta A \propto Q_{\rm encl}$$

➢ To be consistent with Coulomb's law ☛ the proportionality constant is ϵ_0^{-1}

$$\sum_{\substack{\rm closed \\ \rm surface}} E_\perp \, \Delta A = \frac{Q_{\rm encl}}{\epsilon_0}$$

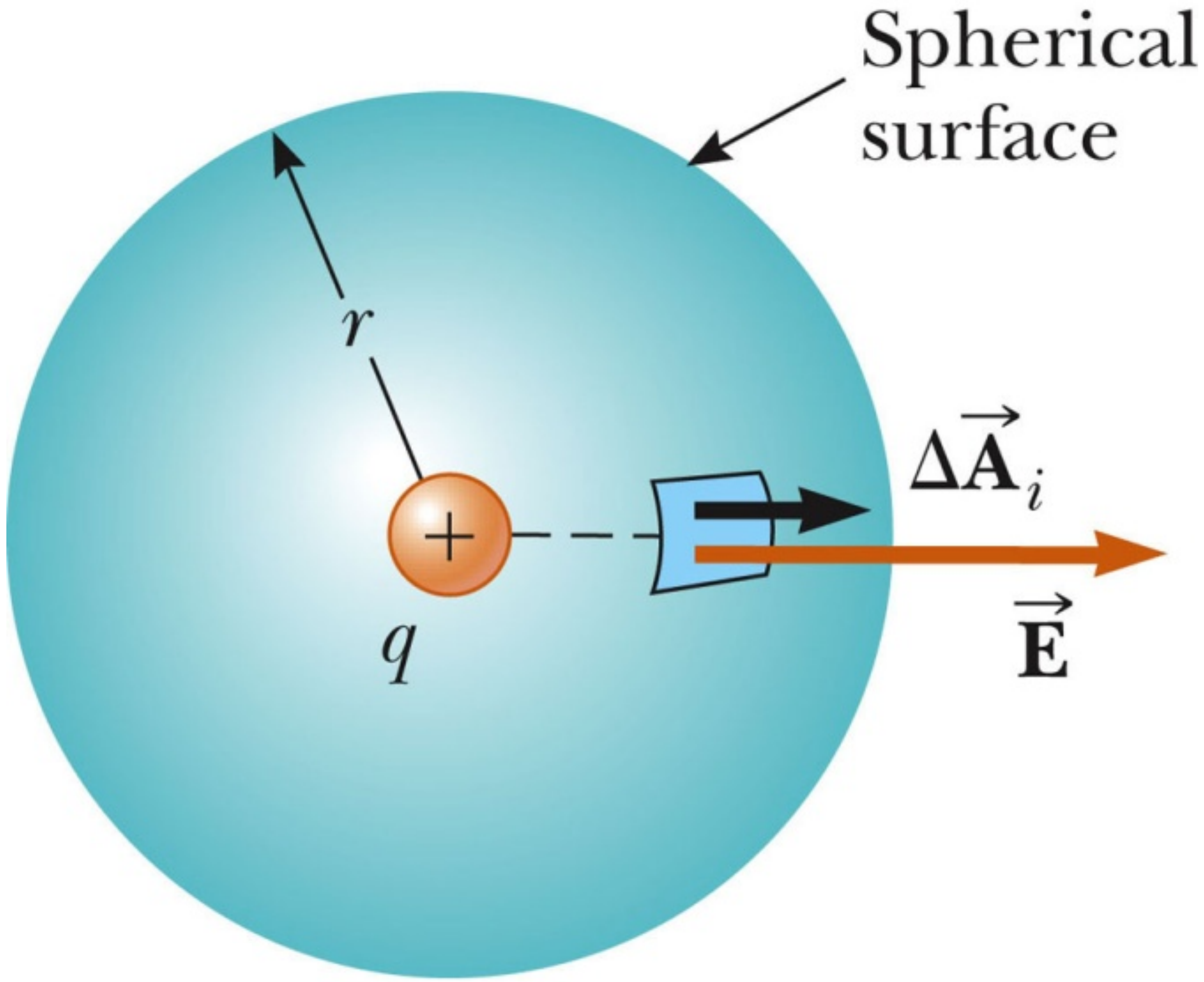
the sum ($\sum_{\substack{\rm closed \\ \rm surface}}$) is over any closed surface and $Q_{\rm encl}$ is the net charge enclosed within that surface

Flux Through a Sphere With a Charge at its Center

➢ A positive point charge q is located at the center of a sphere of radius r

➢ According to Coulomb's Law ☛ magnitude of electric field everywhere on surface of sphere is

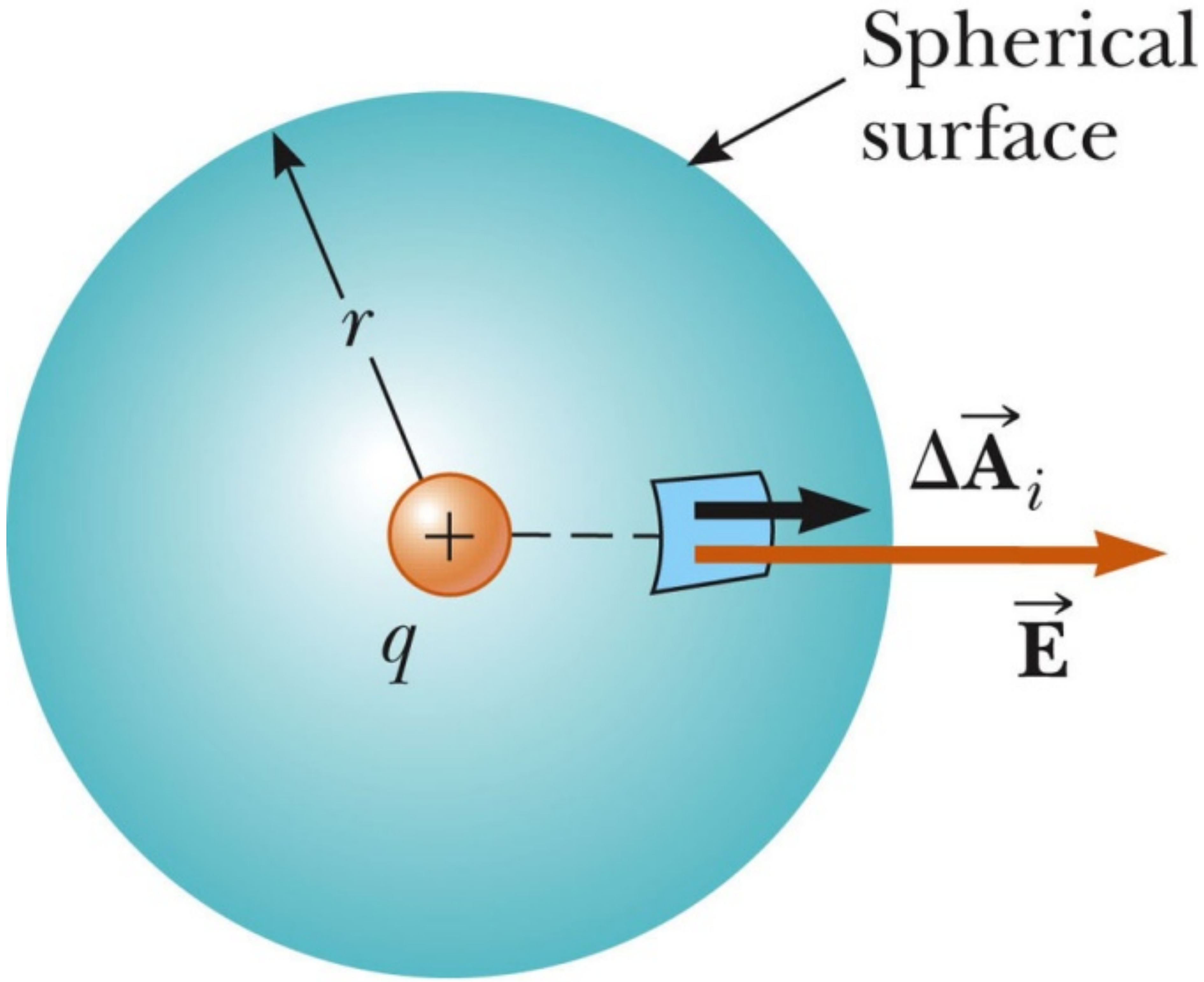
$$E = k\frac{q}{r^2}$$



Flux Through a Sphere With a Charge at its Center

➢ The field lines are directed radially outwards and are perpendicular to surface at every point

$$\Phi_E = \sum_{\text{sphere}} E_\perp \, \Delta A = \sum_{\text{sphere}} E \, \Delta A = E \sum_{\text{sphere}} \Delta A = E 4\pi r^2$$



➢ Combine these two equations we have

$$\Phi_E = E \cdot 4\pi r^2 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}4\pi r^2 = \frac{q}{\epsilon_0}$$

Flux Through a Cube in an Uniform Electric Field

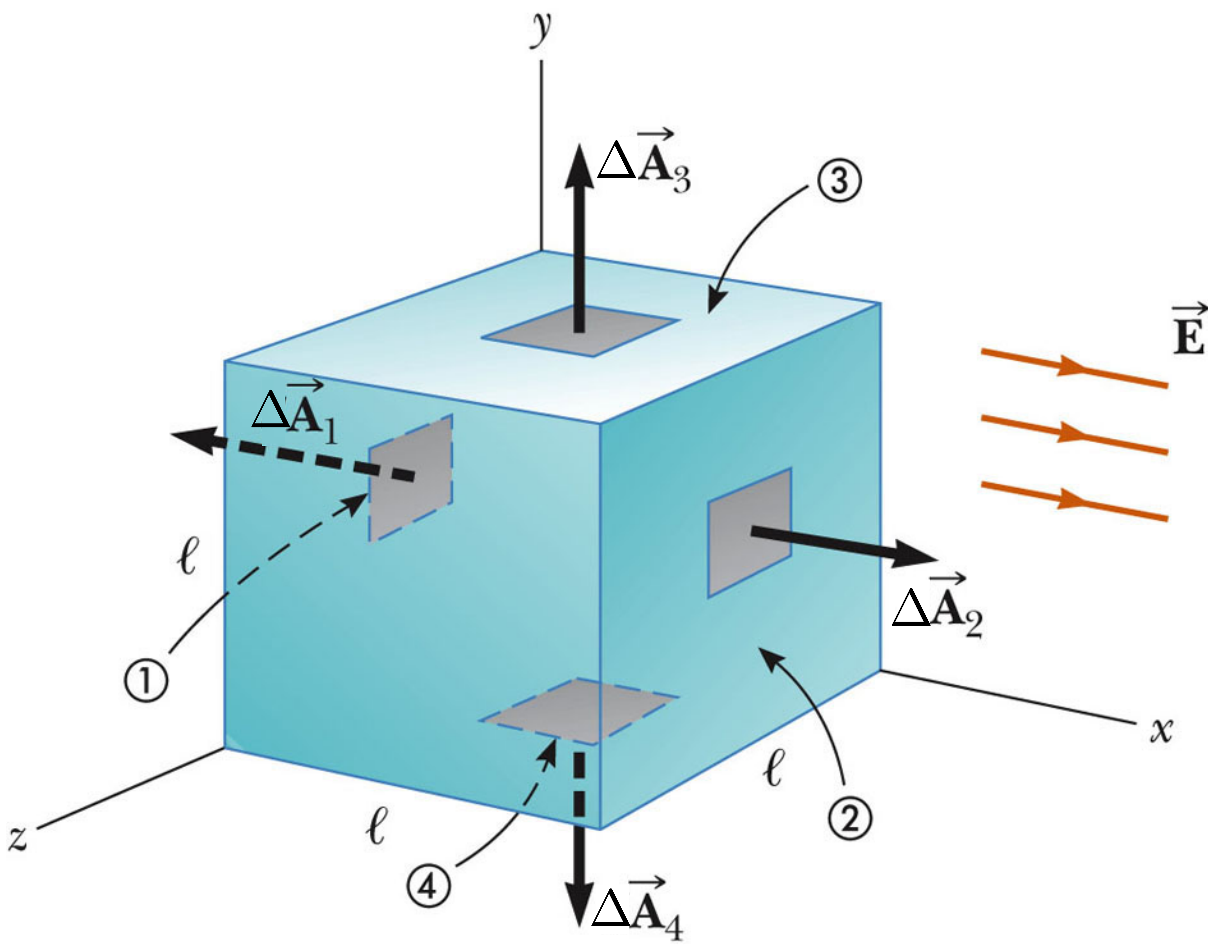
➢ Field lines that pass through surfaces 1 and 2 perpendicularly and are parallel to other four surfaces

➢ For side 1 ☛ $\Delta\Phi_E = -E\ell^2$

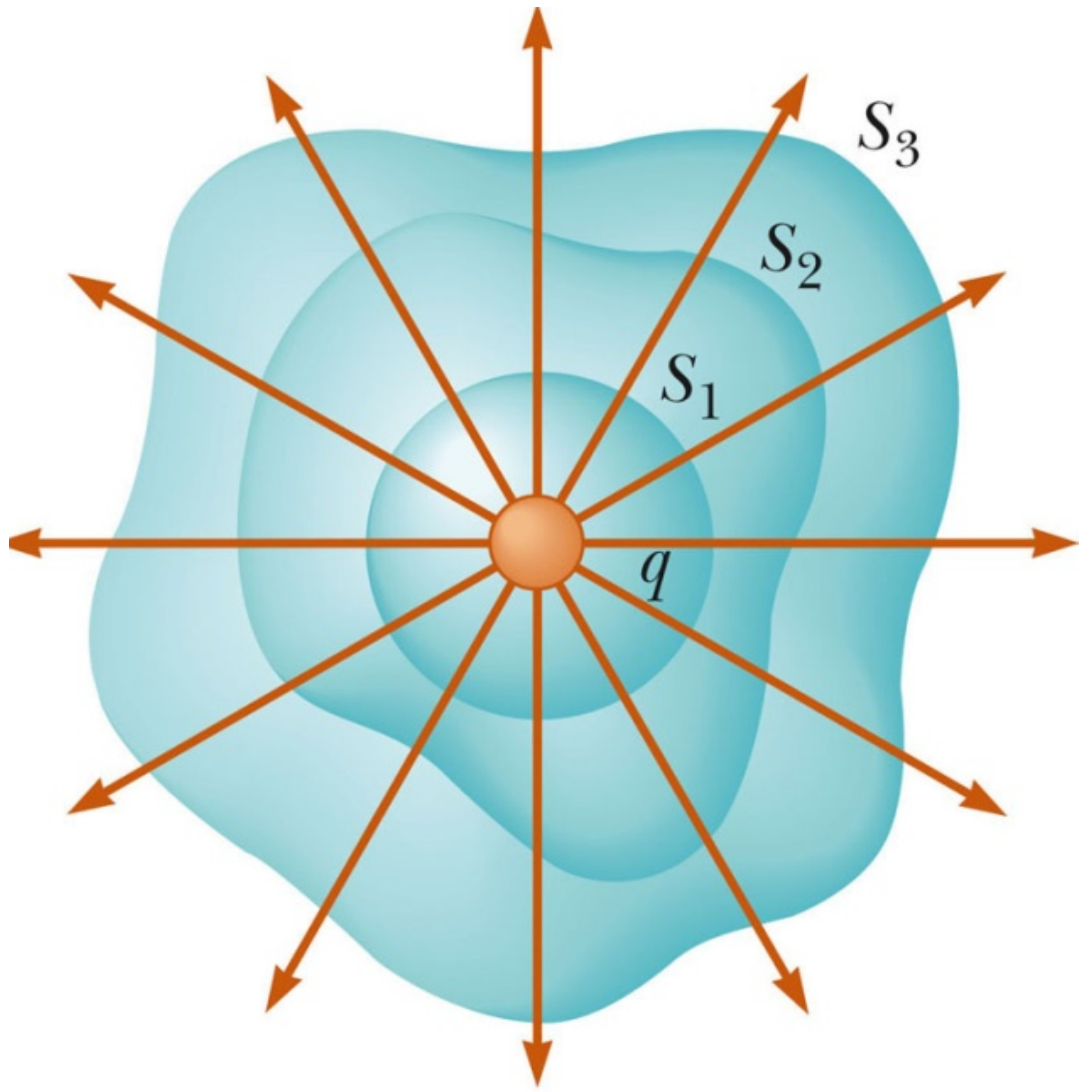
➢ For side 2 ☛ $\Delta\Phi_E = E\ell^2$

➢ For others sides ☛ $\Delta\Phi_E = 0$

➢ Therefore ☛ $\Phi_E = 0$



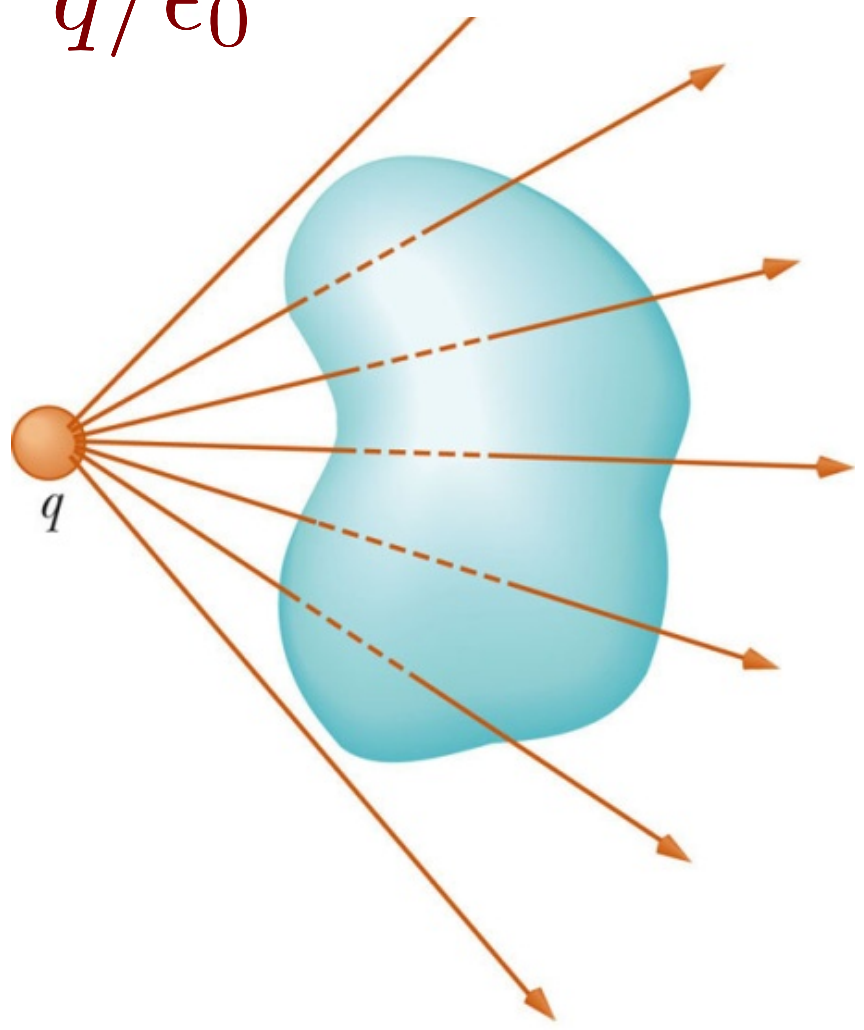
Gaussian Surface & Gauss's Law



- ➢ You choose a **closed surface** and call it a Gaussian Surface
- ➢ This Gaussian Surface can be any shape
- ➢ It may or may not encoles charges
- ➢ Gauss's Law state

Next flux through any closed surface surrounding a charge q is given by q/ϵ_0 and is independent of shape of that surface

$$\sum_{\substack{\text{closed}\\ \text{surface}}} E_{\perp}\, \Delta A = \frac{Q_{\text{encl}}}{\epsilon_0}$$



Applying Gauss Law

To use Gauss law ☛ choose a Gaussian surface over which surface $\sum$ can be simplified and electric field determined

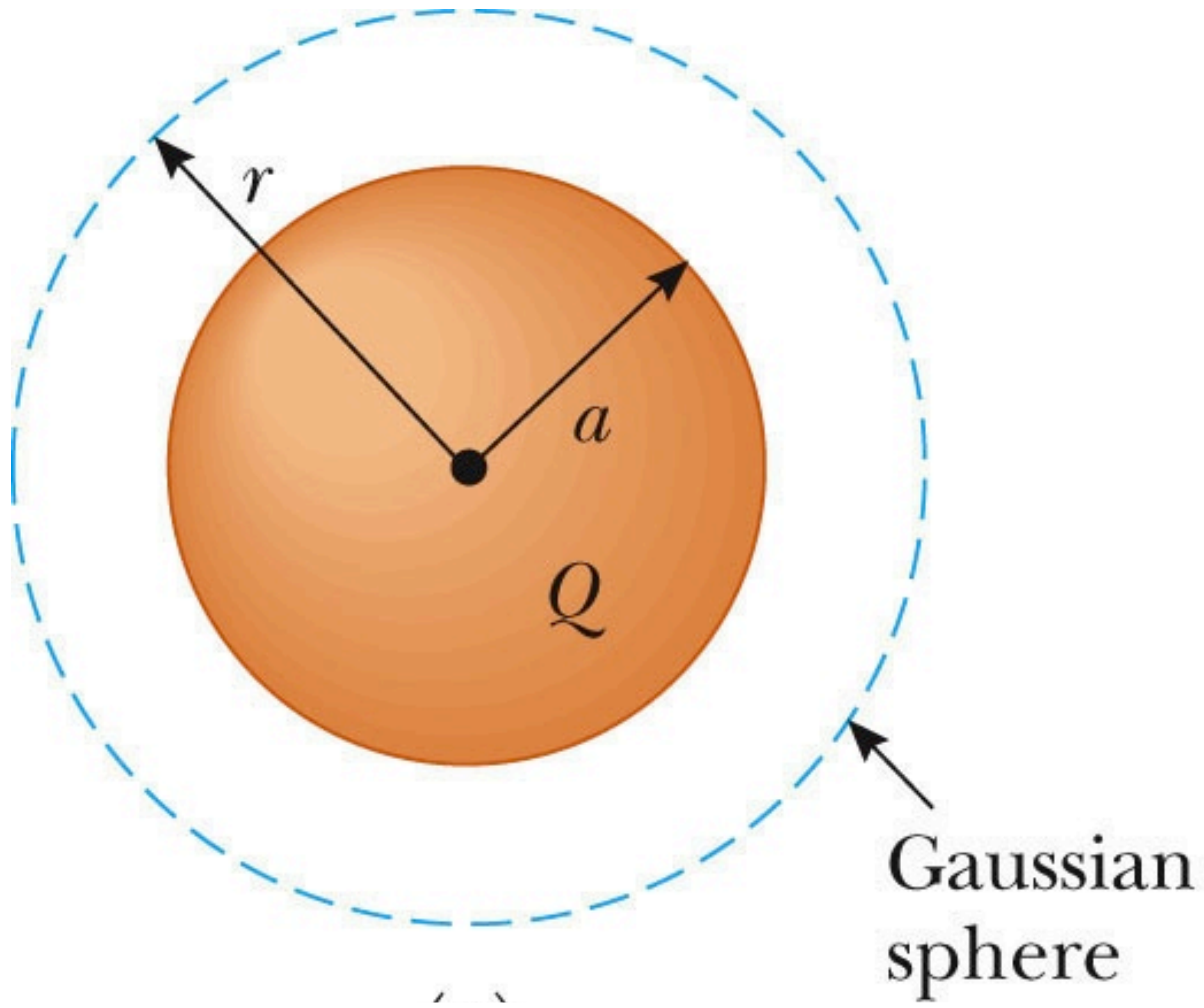
Take advantage of symmetry

Remember

- ☛ **Gaussian surface is a surface you choose**
- ☛ **it does not have to coincide with a real surface**

Field Due to a Spherically Symmetric Even Charge Distribution

- ➢ Field must be different inside $(r < a)$ and outside $(r > a)$ of sphere
- ➢ For $r > a$ select a sphere as Gaussian surface with radius r and concentric to original sphere
- ➢ Because of this symmetry ☛ electric field direction must be radially along r and at a given r ☛ field's magnitude is a constant



Can you write down mathematical expression based on above reasoning?

Field Due to a Spherically Symmetric Even Charge Distribution

➢ Field must be different inside $(r < a)$ and outside $(r > a)$ of sphere

➢ For $r > a$ select a sphere as Gaussian surface with radius r and concentric to original sphere

➢ Because of this symmetry ☛ electric field direction must be radially along r

and at a given r ☛ field's magnitude is a constant

E is constant at a given r

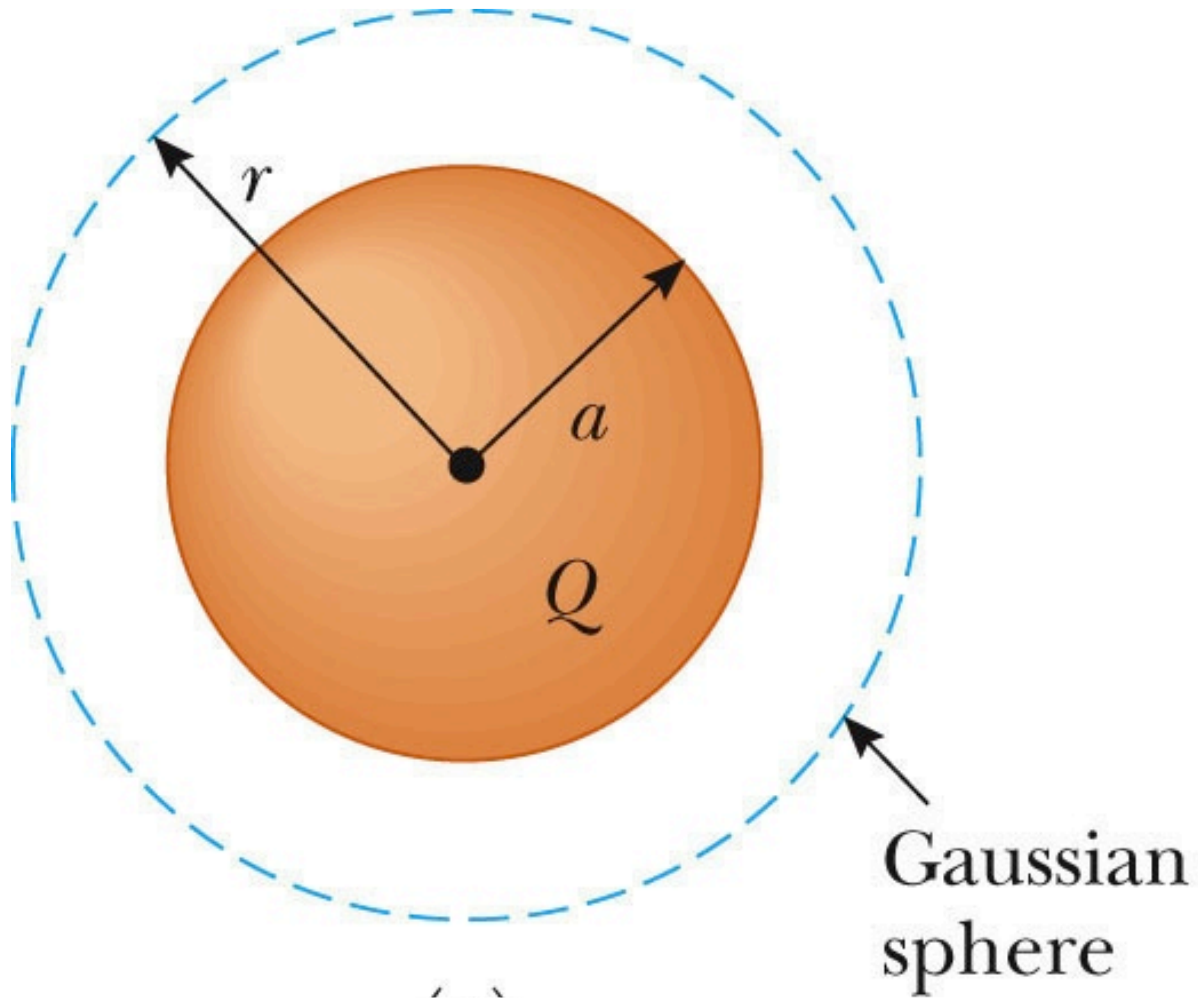
$$\Phi_E = \sum_{\text{sphere}} E_\perp \, \Delta A = \sum_{\text{sphere}} E \, \Delta A = E \sum_{\text{sphere}} \Delta A = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

Gauss Law

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} = k\frac{Q}{r^2}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

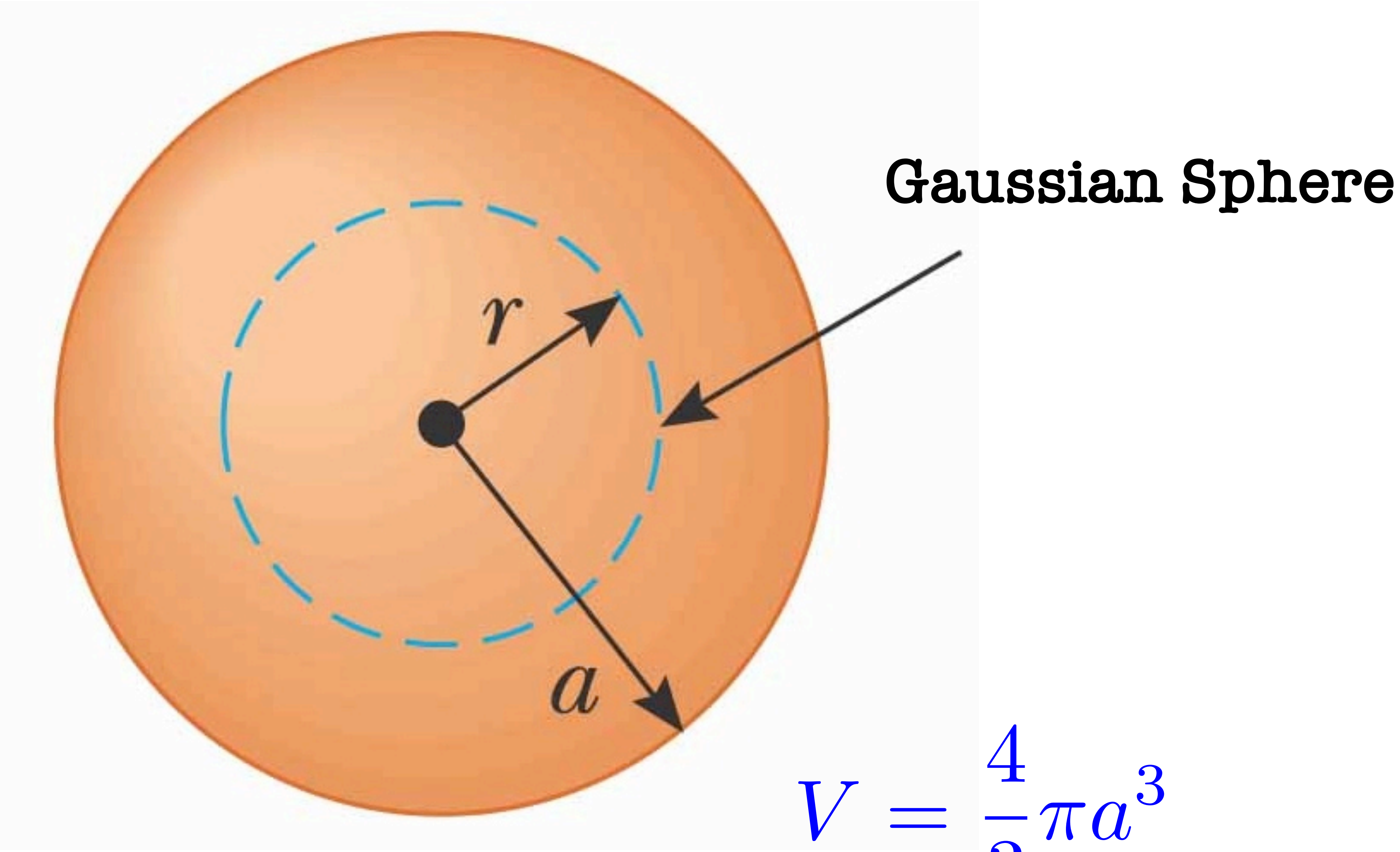
As if the charge is a point charge Q



Field Inside Sphere

- For $r < a$ select a sphere as Gaussian surface
- All arguments are same as for $r > a$
- The only difference is here $Q_{\text{encl}} < Q$
- Find out that $Q_{\text{encl}} = Q(r/a)^3$

How?



$$V = \frac{4}{3}\pi a^3$$

$$\Phi_E = \sum_{\text{sphere}} E_\perp \, \Delta A = E \cdot 4\pi r^2 = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$\rho = \frac{Q}{V} \Rightarrow Q_{\text{encl}} = \rho \cdot \frac{4}{3}\pi r^3$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{encl}}}{r^2} = k\frac{1}{r^2}\frac{r^3}{a^3}Q = k\frac{Q}{a^3}r$$

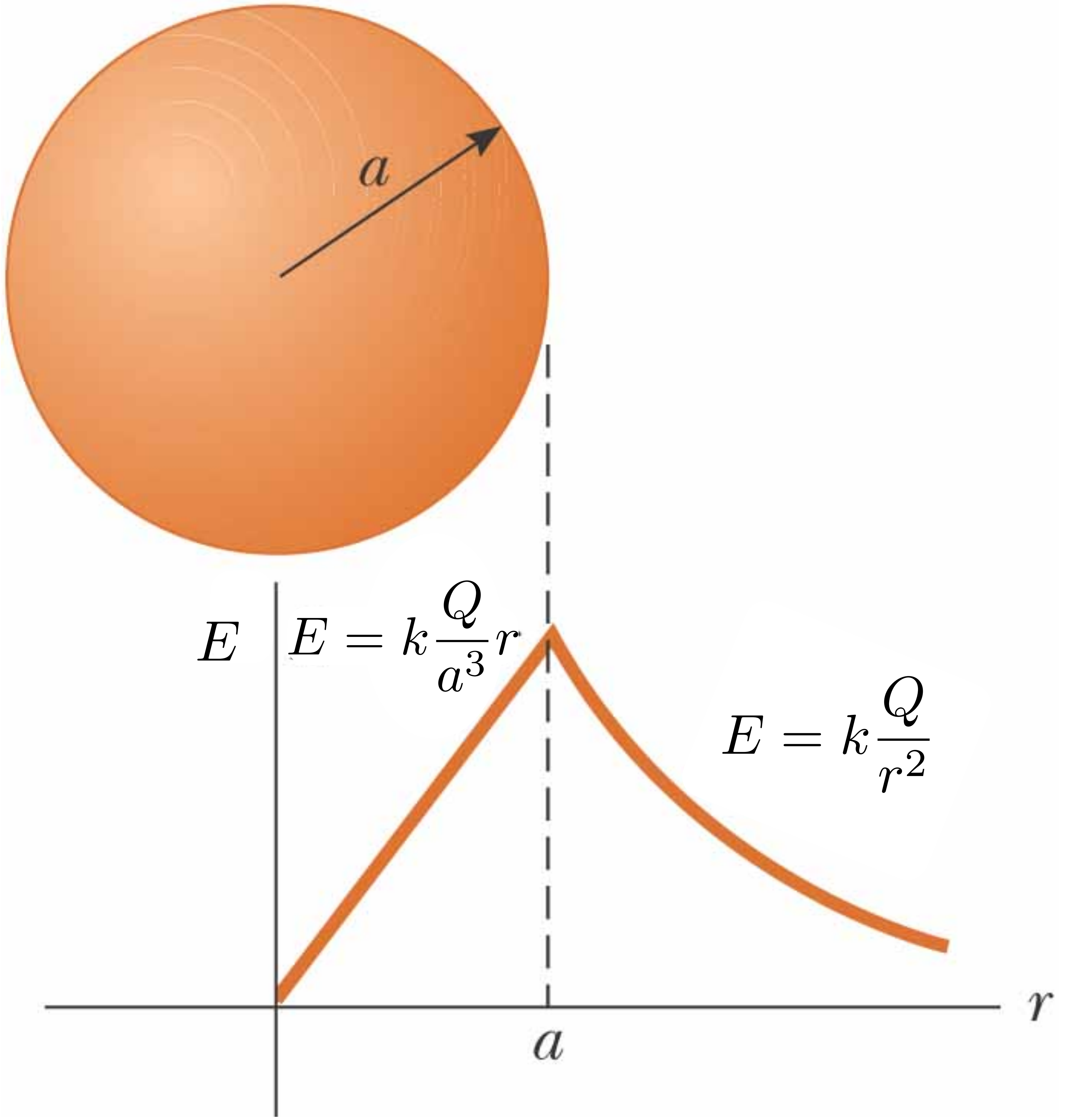
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{a^2}\vec{r}$$

Increase linearly with r not with $1/r^2$

Plot Results (Assume Positive Q)

➢ Inside sphere E varies linearly with r

E approches 0 as r approches 0



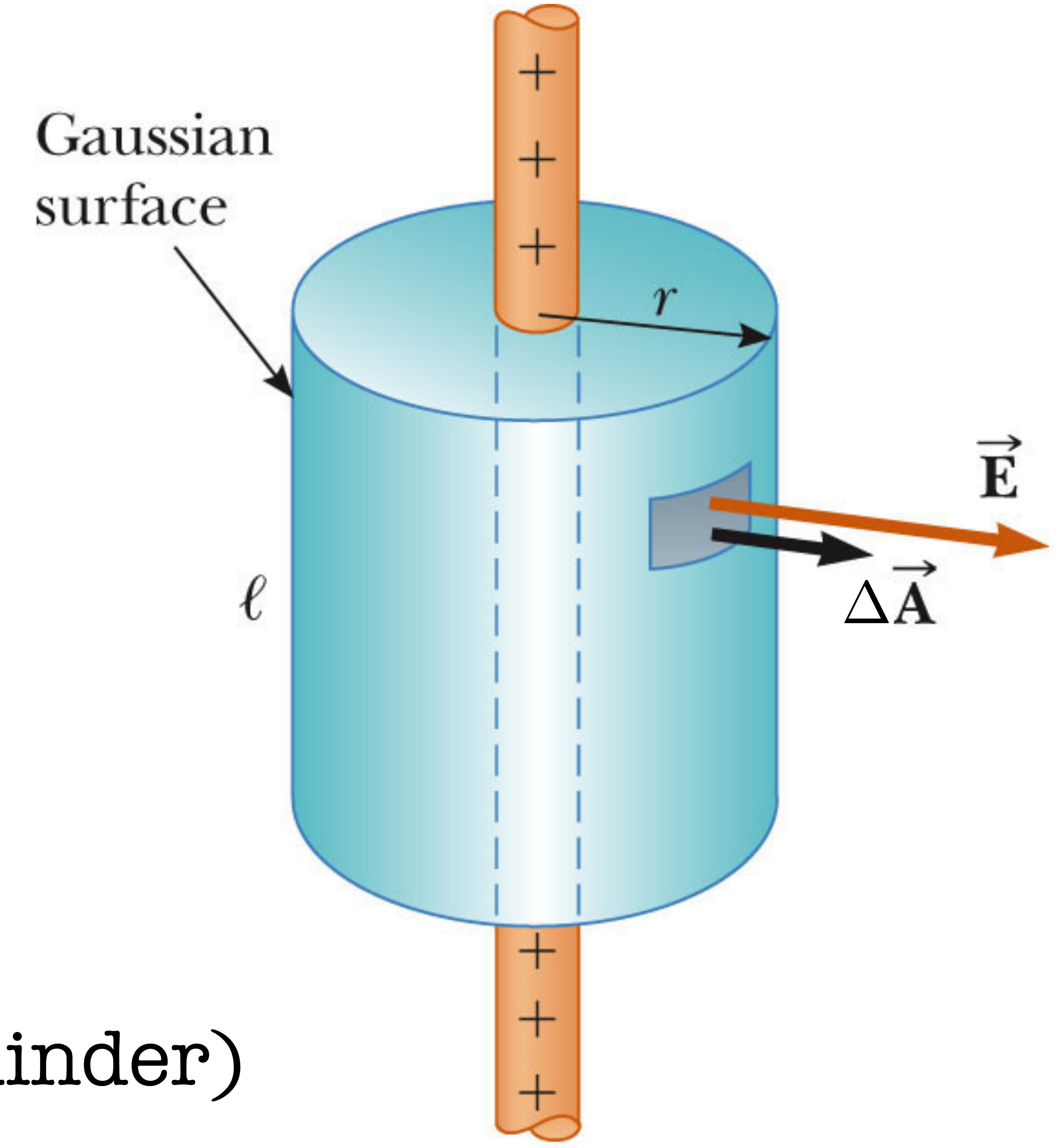
➢ Field outside sphere is equivalent to that of a point charge located at center of sphere

Field at a Distance from a Straight Line of Charge

- Select a cylinder as Gaussian surface
- Cylinder has a radius of r and a length of ℓ
- $\vec{E}$ is constant in magnitude and parallel to surface

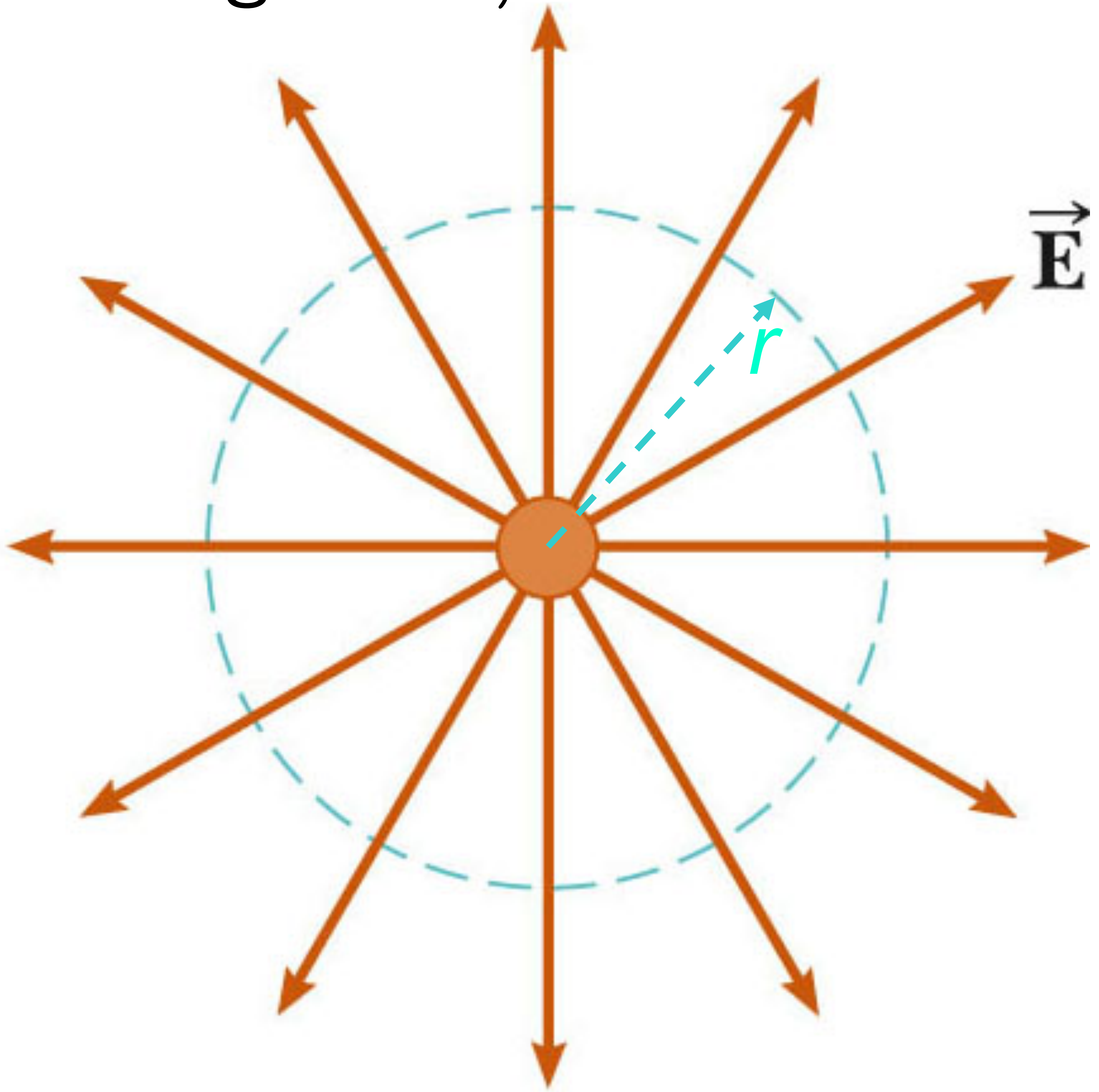
  (direction of a surface is its normal!)

  at every point on curved part of surface (body of cylinder)



Calculate Flux

- Because of this line symmetry, end view illustrates more clearly that field is parallel to curved surface, and constant at a given r, so flux is $\Phi_E = E \cdot 2\pi r \ell$



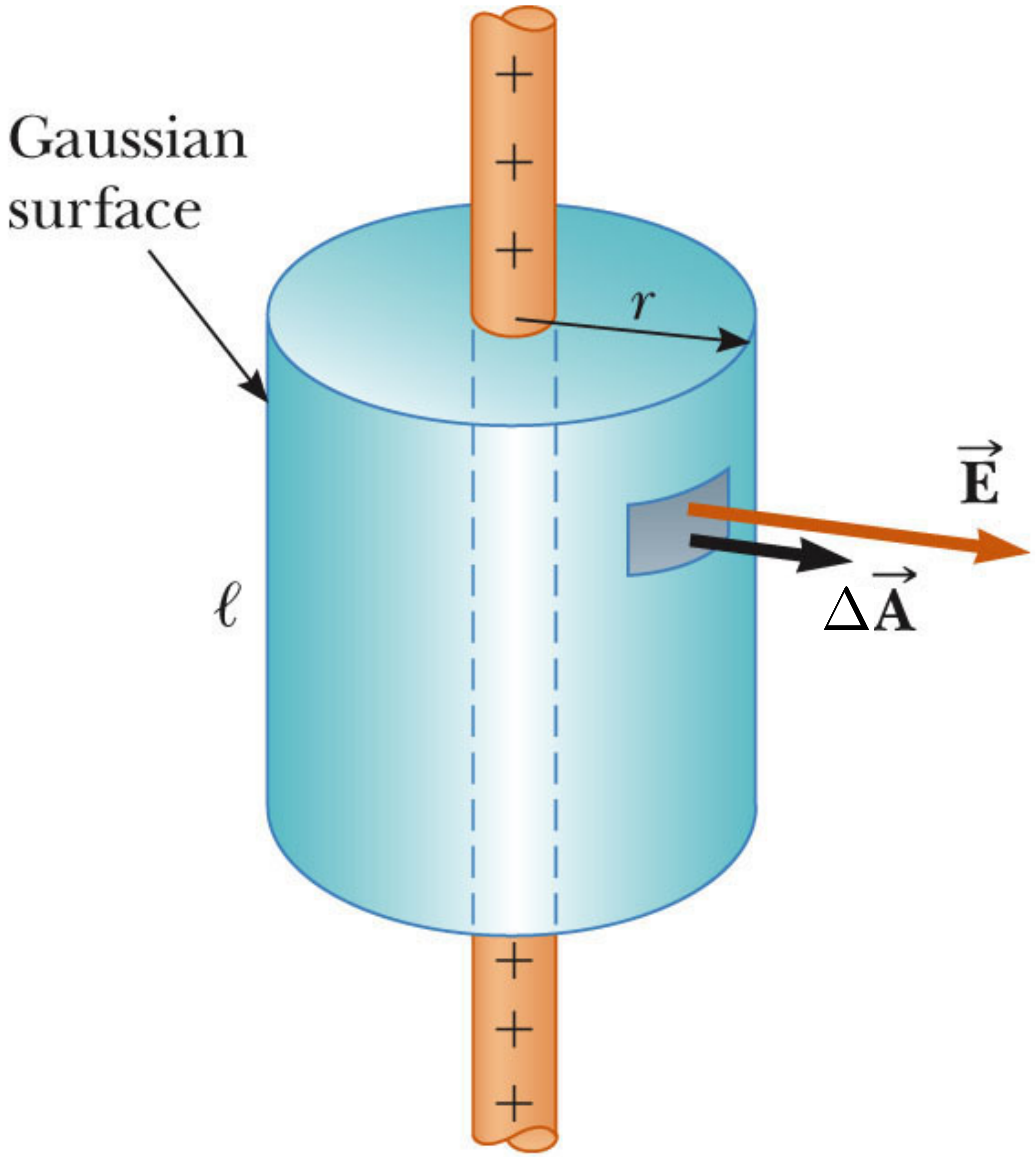
- Flux through ends of cylinder is 0 since field is perpendicular to these surfaces

Electric Field from Gauss Law

$$\Phi_E = \sum_{\text{cylinder}} E_\perp \, \Delta A = \frac{q}{\epsilon_0}$$

$$E \cdot 2\pi r \ell = \frac{\lambda \ell}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = 2k\frac{\lambda}{r}$$



One can change thin wire into a rod as we did in sphere case and find electric field inside & outside of rod

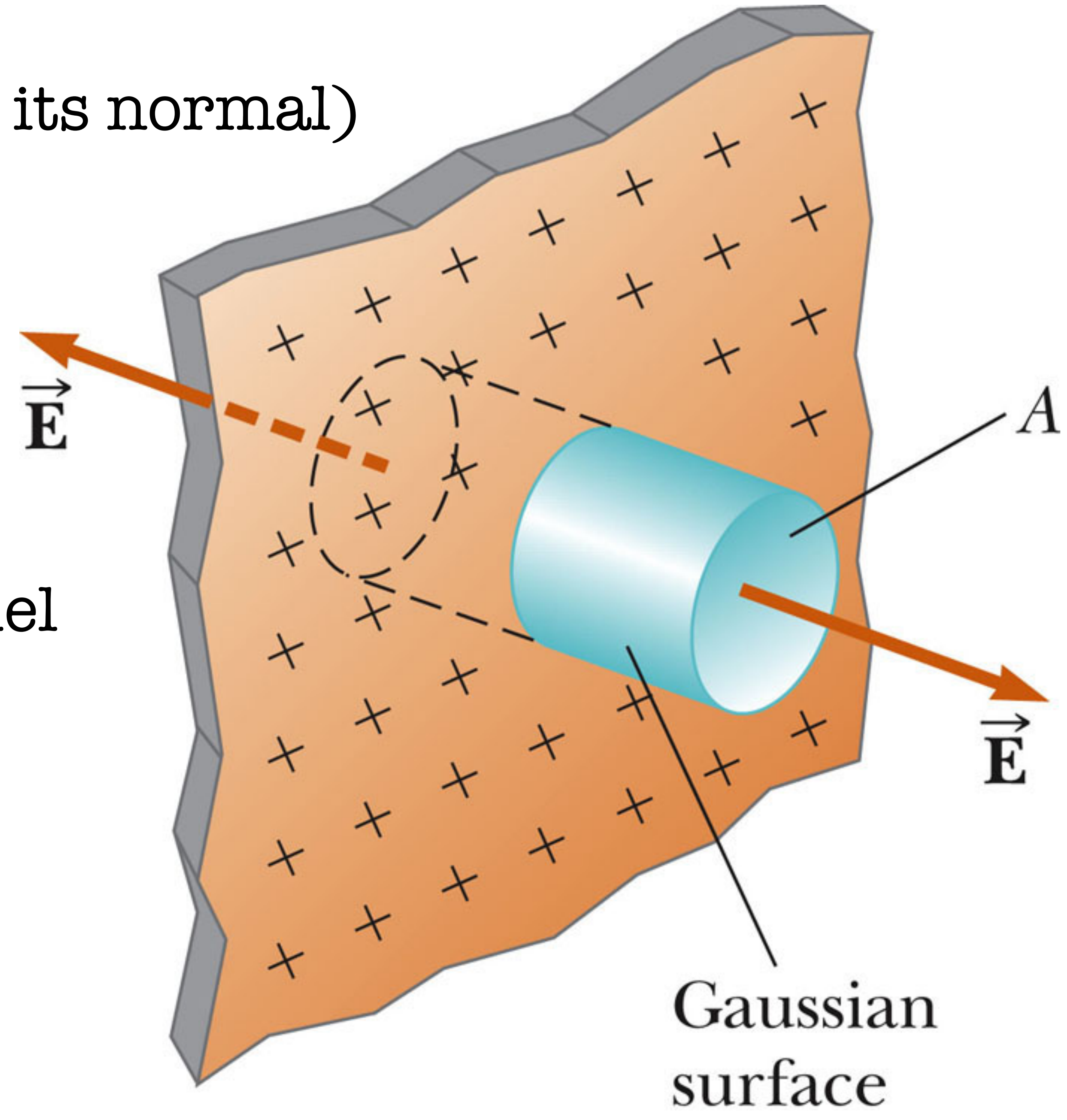
Field Due to an Infinitely Large Plane of Charge

➢ Argument about electric field ☛

Because plane is infinitely large, any point can be treated as center point of plane

so at that point $\vec{E}$ must be parallel to plane direction (again this is its normal)

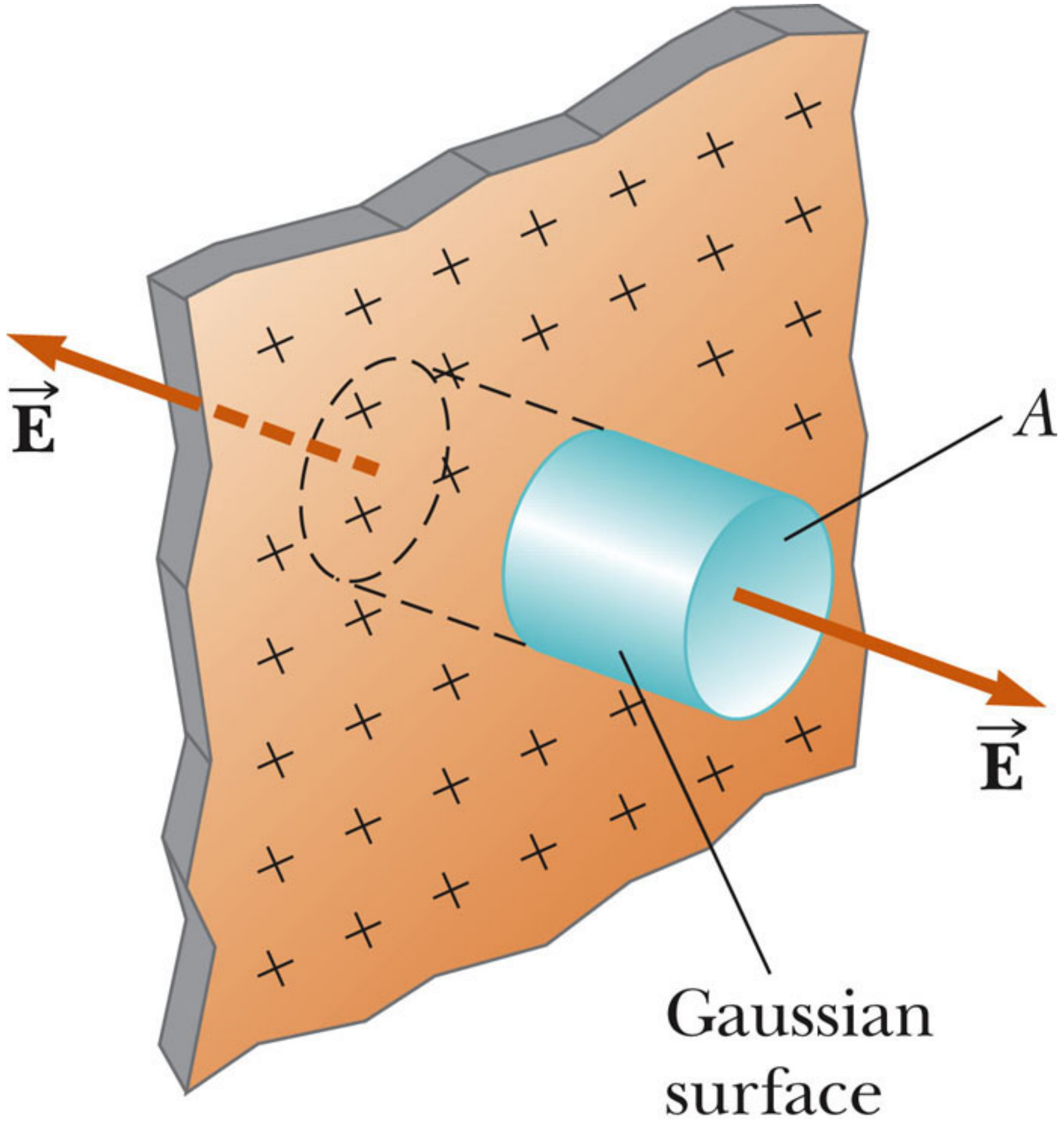
and must have same magnitude at all points equidistant from plane



➢ Choose Gaussian surface to be a small cylinder whose axis is parallel to plane 's direction (third time, this is normal of plane)

Find Out Flux

- $\vec{E}$ is parallel to ends ☛ flux through each end of cylinder is EA and total flux is $2EA$



- $\vec{E}$ is perpendicular to curved surface direction flux through this surface is 0 because cos(90°) = 0

Electric Field from Gauss Law

- Total charge in surface is $Q = \sigma A$
- Applying Gauss's law

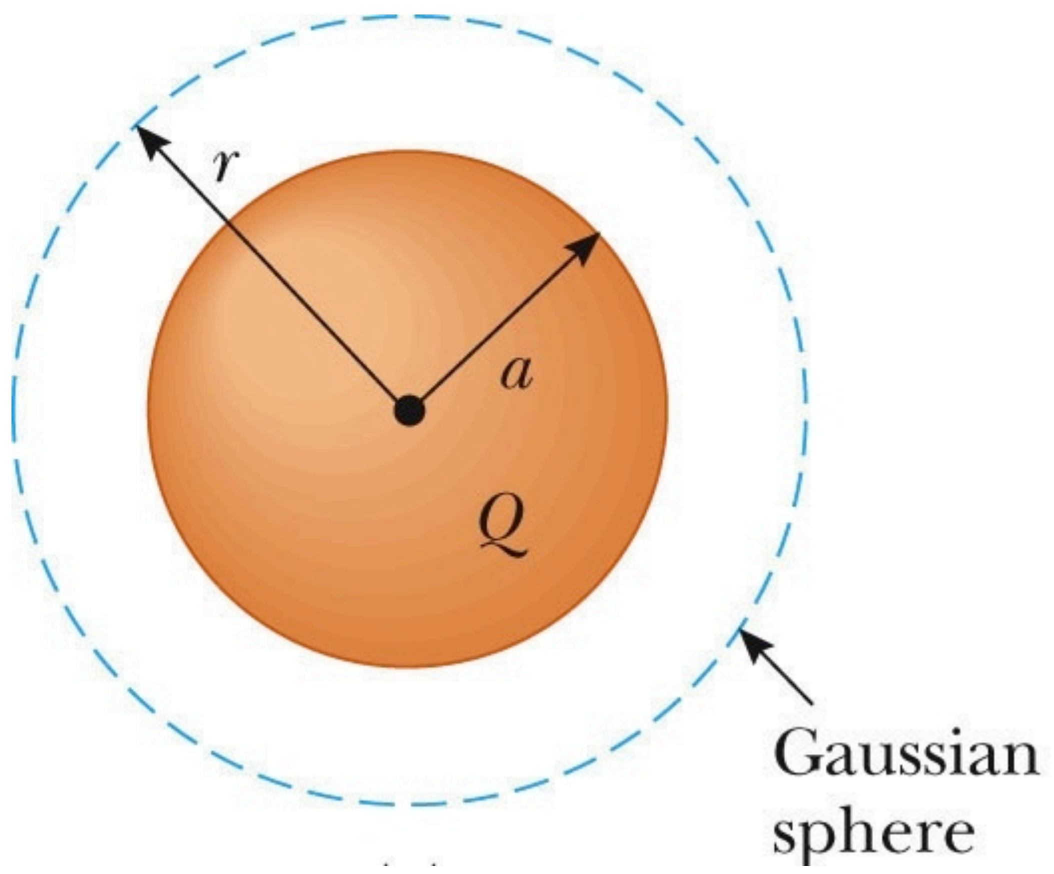
$$\Phi_E = 2EA = \frac{\sigma A}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{2\epsilon_0}$$

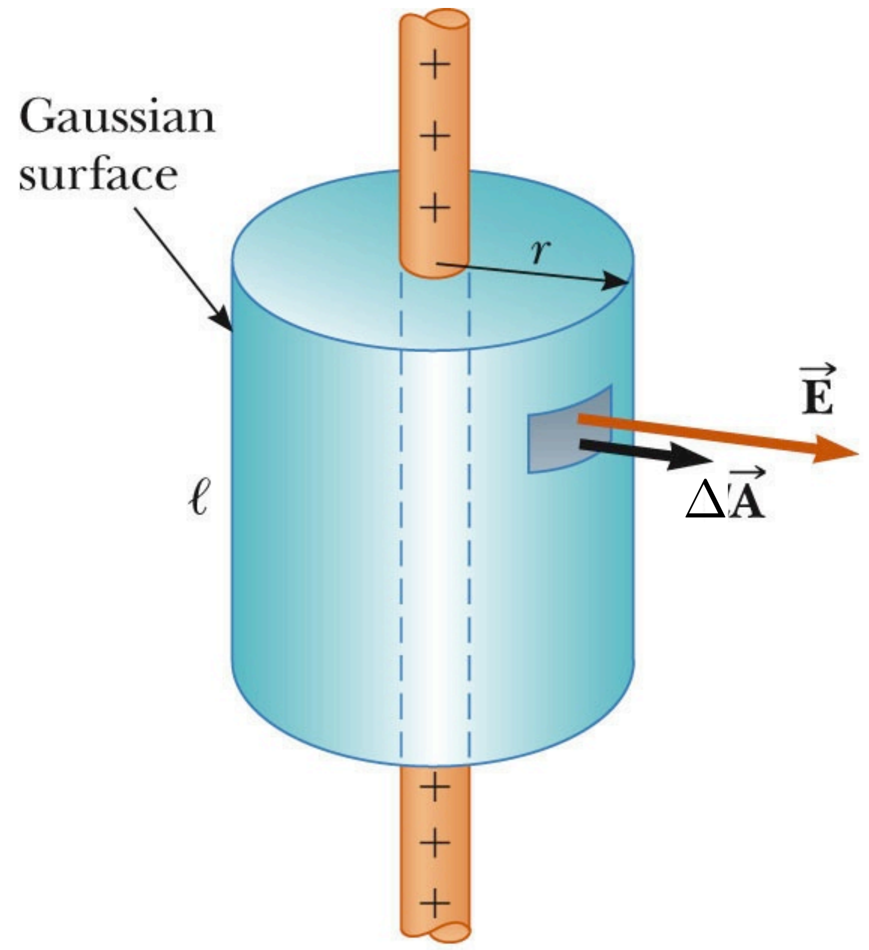
- Note, this does not depend on r ☛ distance from point of interest to charge plane

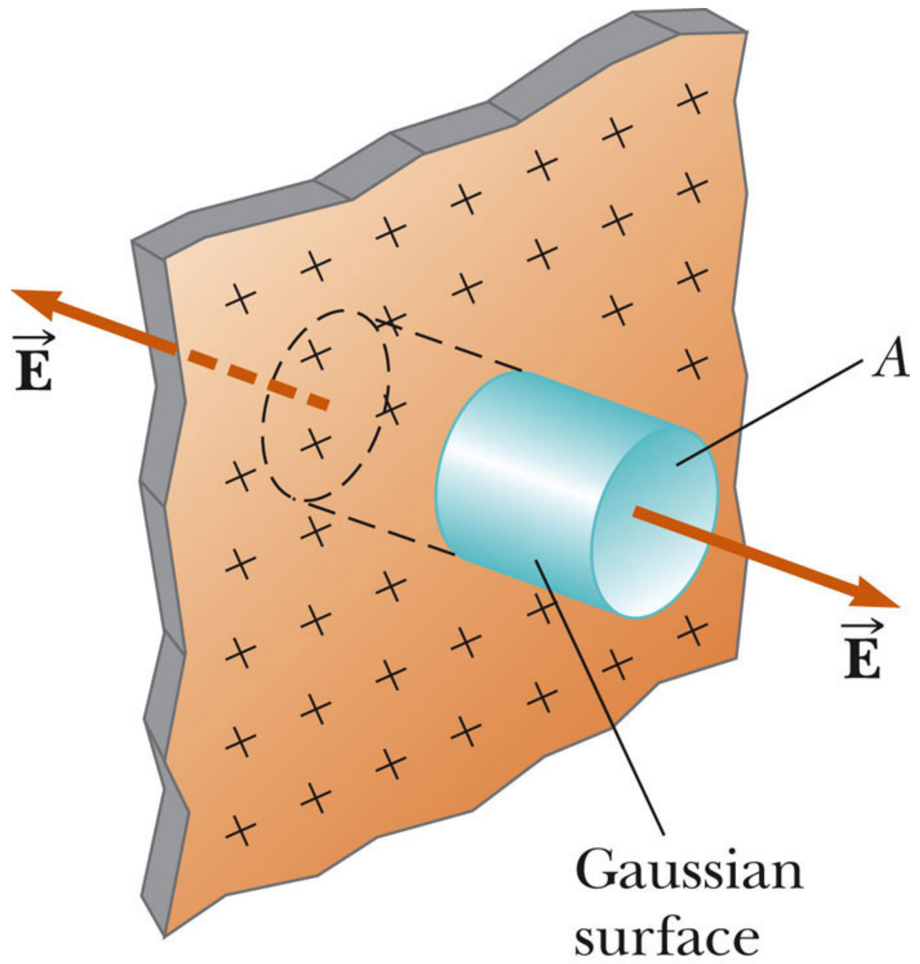
WHY?

- Therefore, field is uniform everywhere

To Summarize
3 Types of Gauss Law Problems







$$E = k\frac{Q}{a^3}r \quad ☛ \quad r < a$$

$$E = k\frac{Q}{r^2} \quad ☛ \quad r \geq a$$

$$E = 2k\frac{\lambda}{r}$$

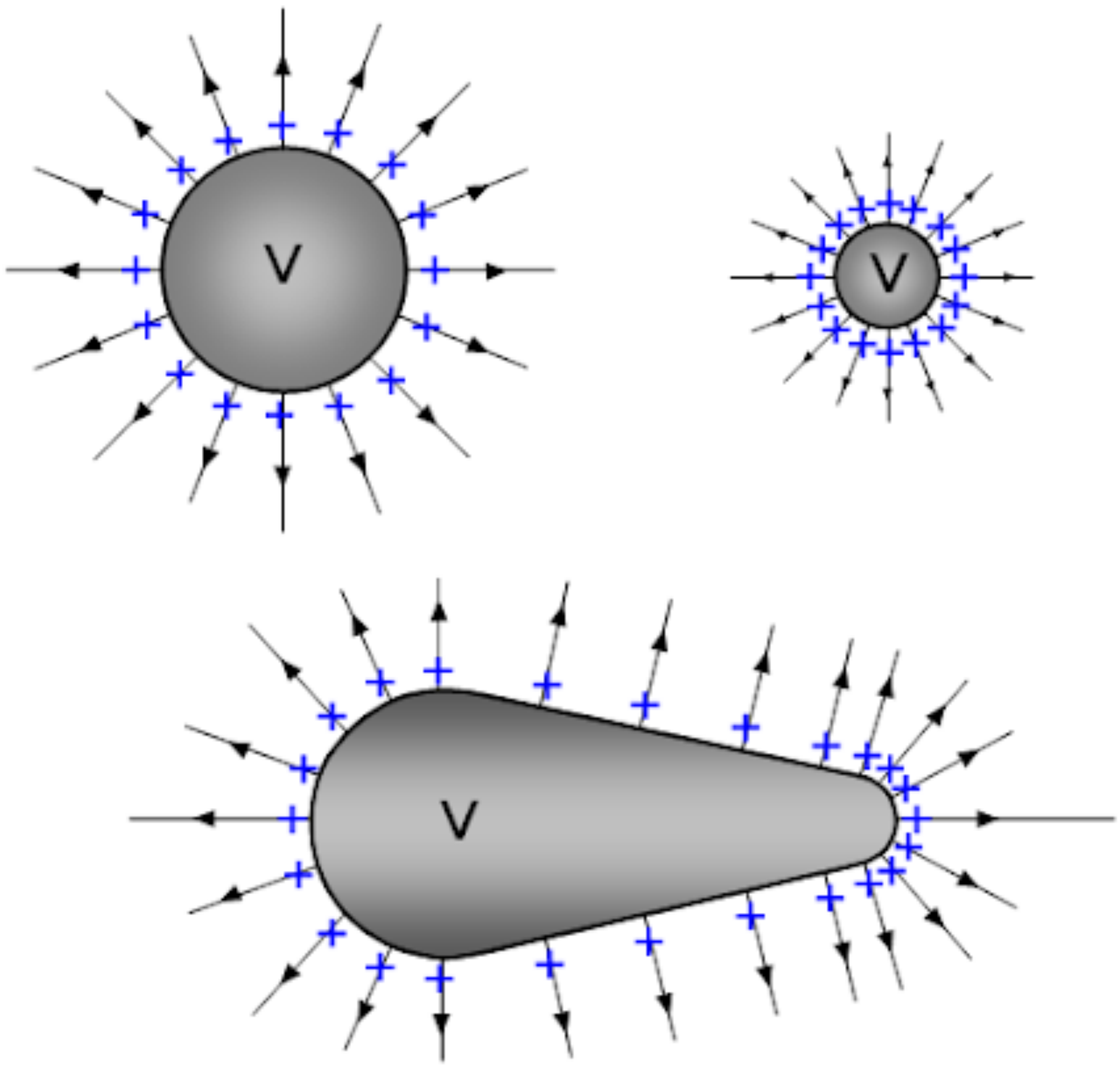
$$E = \frac{\sigma}{2\epsilon_0}$$

Conductors in electrostatic equilibrium

- ➢ Electrical conductors contain charges (electrons) that are not bound to any atom and therefore are free to move about within the material
- ➢ When there is no net motion of charge within a conductor ☛ the conductor is in electrostatic equilibrium
- ➢ A conductor in electrostatic equilibrium has the following properties:

1. Electric field is zero everywhere inside conductor
2. If an isolated conductor carries a charge ☛ charge resides on its surface
3. Electric field just outside a charged conductor is perpendicular to surface of conductor and has a magnitude σ / ϵ_0 σ ☛ surface charge density at that point!
4. On an irregularly shape conductor ➢ surface charge density is greates at locations where radius of curvature of surface is smallest

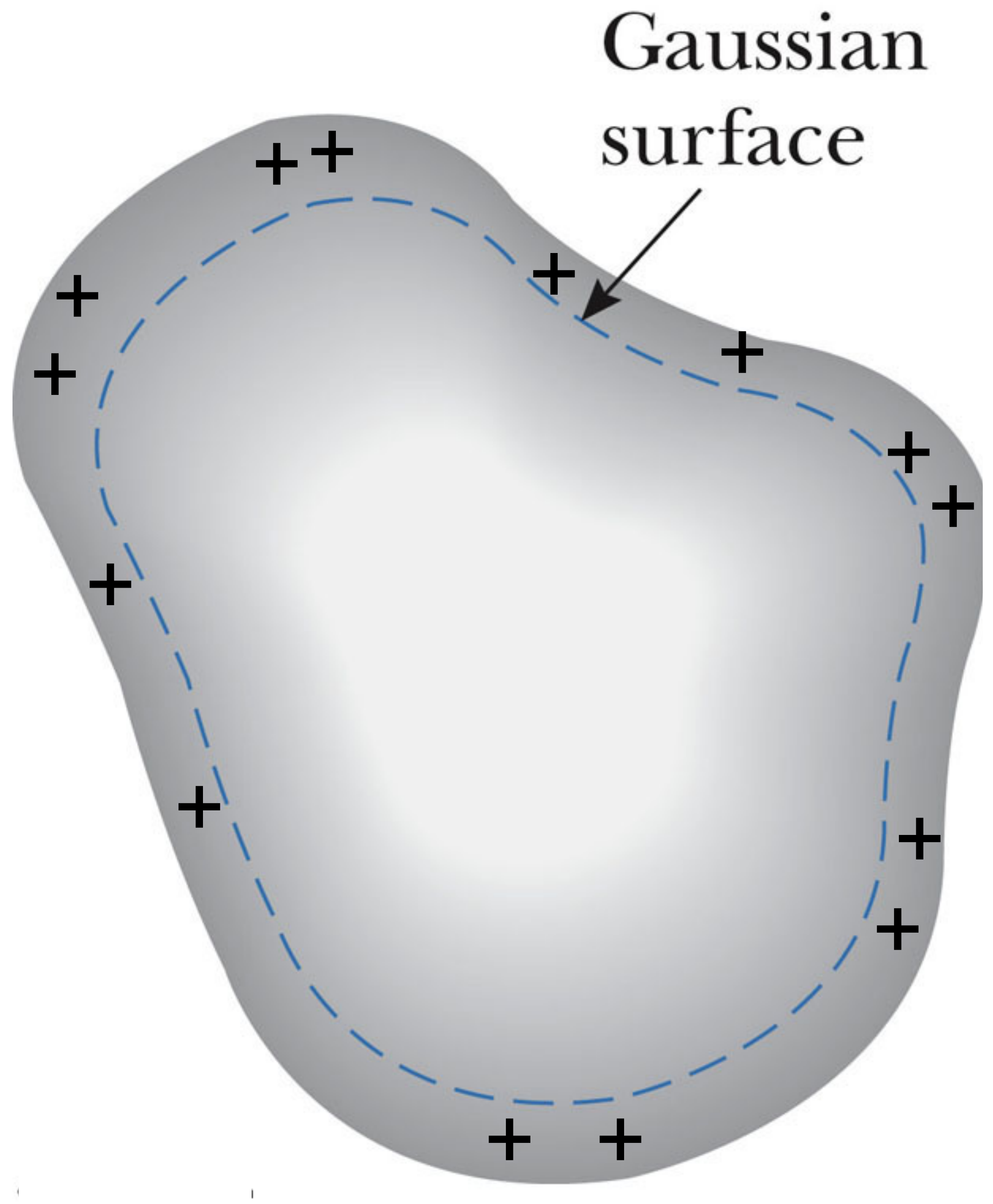
Charge distribution in different volumes



Property 2: For a charged conductor, charge resides on surface, and field inside conductor is stil zero

➢ Charges (have to be the same sign, why?) repel and move away from each other until they reach the surface and can no longer move out: charge resides only on the surface because of Coulomb's Law

➢ Choose a Gaussian surface inside but close to the actual surface



➢ Since there is no net charge inside this Gaussian surface,

there is no net flux through it.

➢ Because the Gaussian surface can be any where inside the volume

and as close to the actual surface as desired, the electric field inside this volume is zero anywhere

Property 3: Field's Magnitude and Directios on Surface

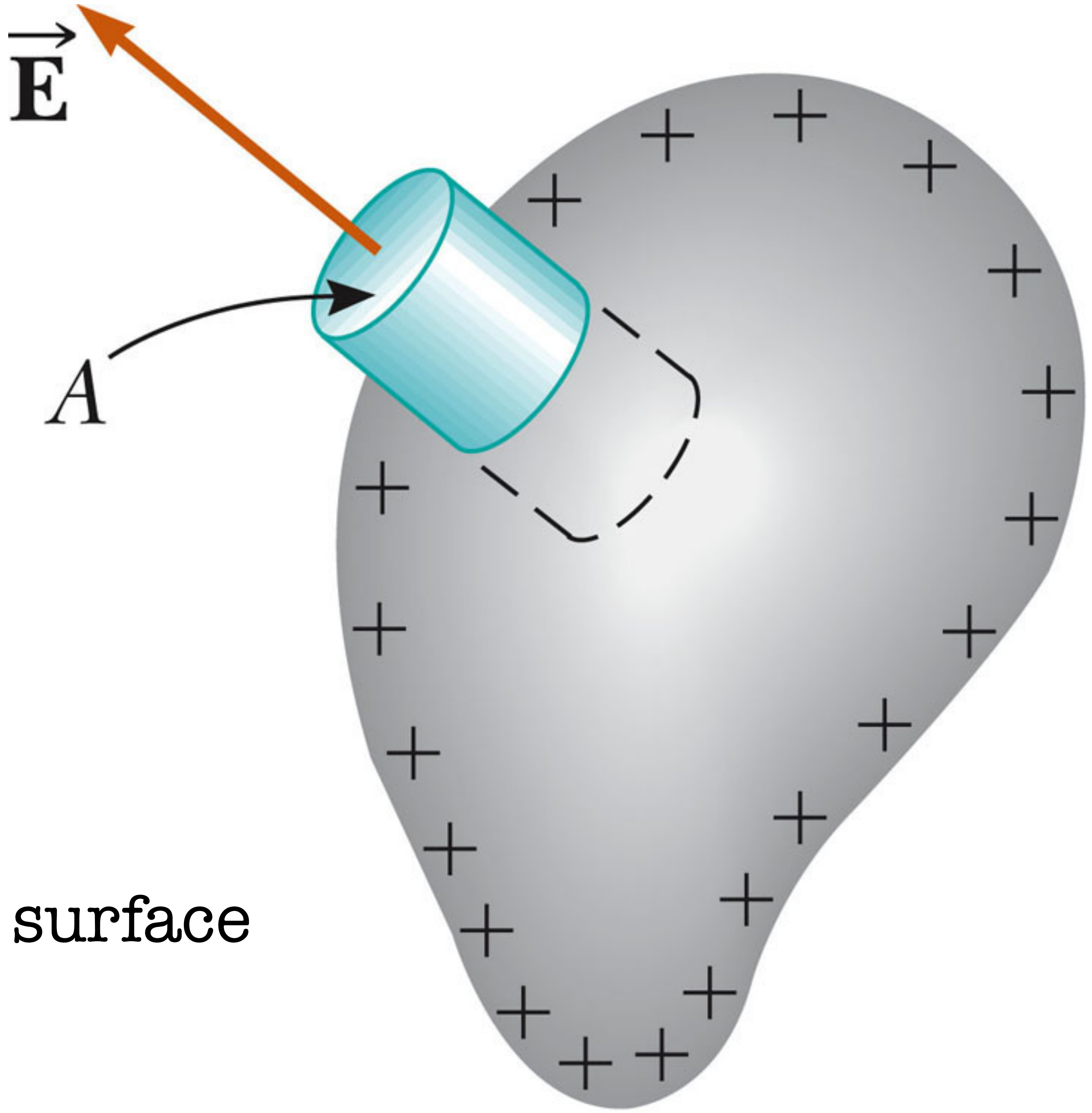
Direction

➢ Choose a cylinder as the gaussian surface

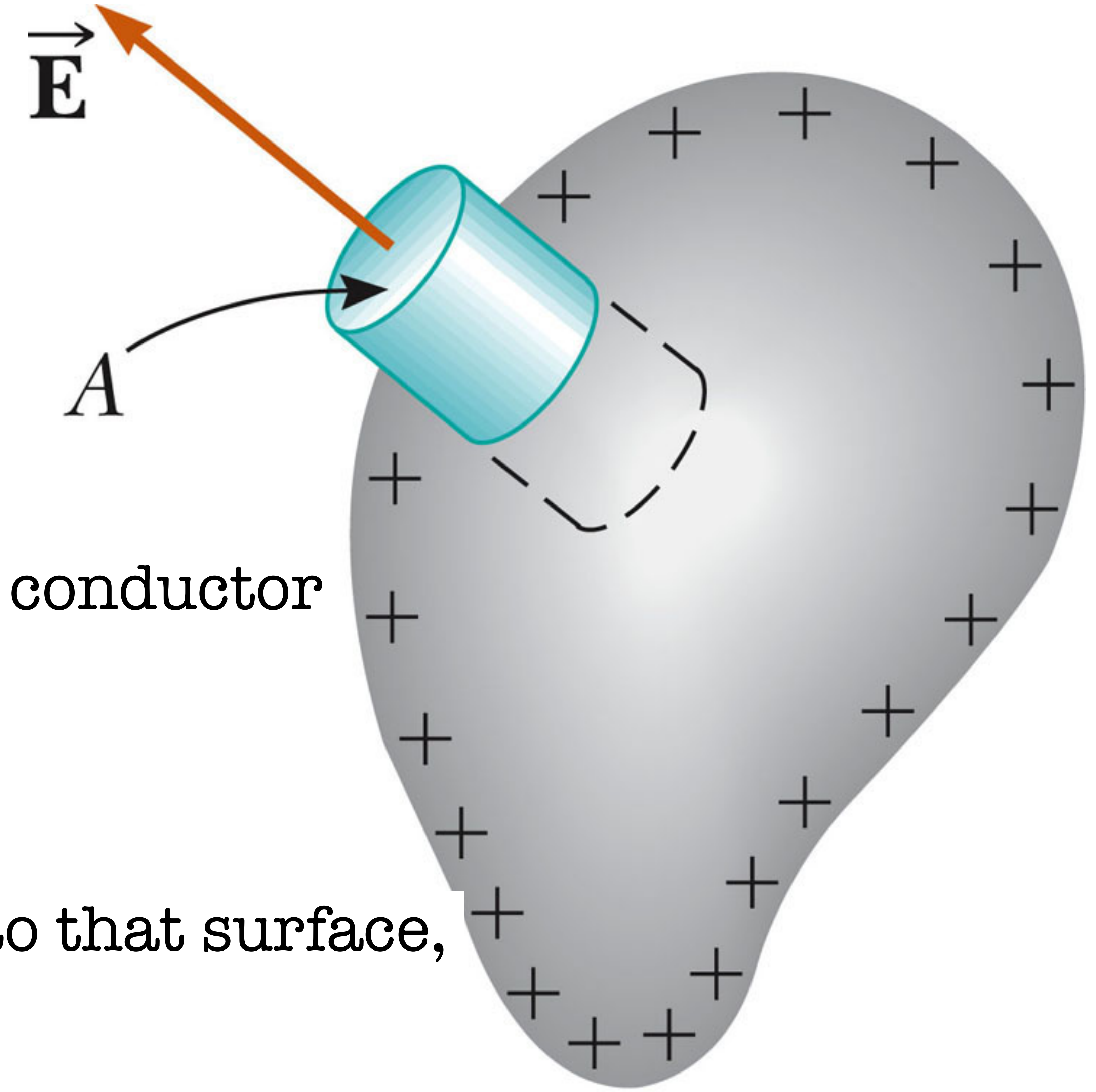
➢ The field must be parallel to the surface (again this is its normal)

✱ If there were an angle $(\theta \neq 0)$, then there were a component $E_\perp$ from $\vec{E}$ and tangent to the surface that would move charges along surface

Then conductor would not be in equilibrium (no charge motions)



Property 3: Field's Magnitude and Directios on Surface



Magnitude

➢ Choose a Gaussian surface as an infinitesimal cylinder with its axis parallel to conductor surface, as shown in figure

➢ Net flux through Gaussian surface is that only through flat face outside conductor

* Field here is parallel to surface

* Field on all other surfaces of Gaussian cylinder is either perpendicular to that surface, or zero

➢ Applying Gauss's law, we have

$$\Phi_E = EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{\epsilon_0}$$

Another Example : Electric Field Generated by a Conducting Sphere and a Conducting Shell

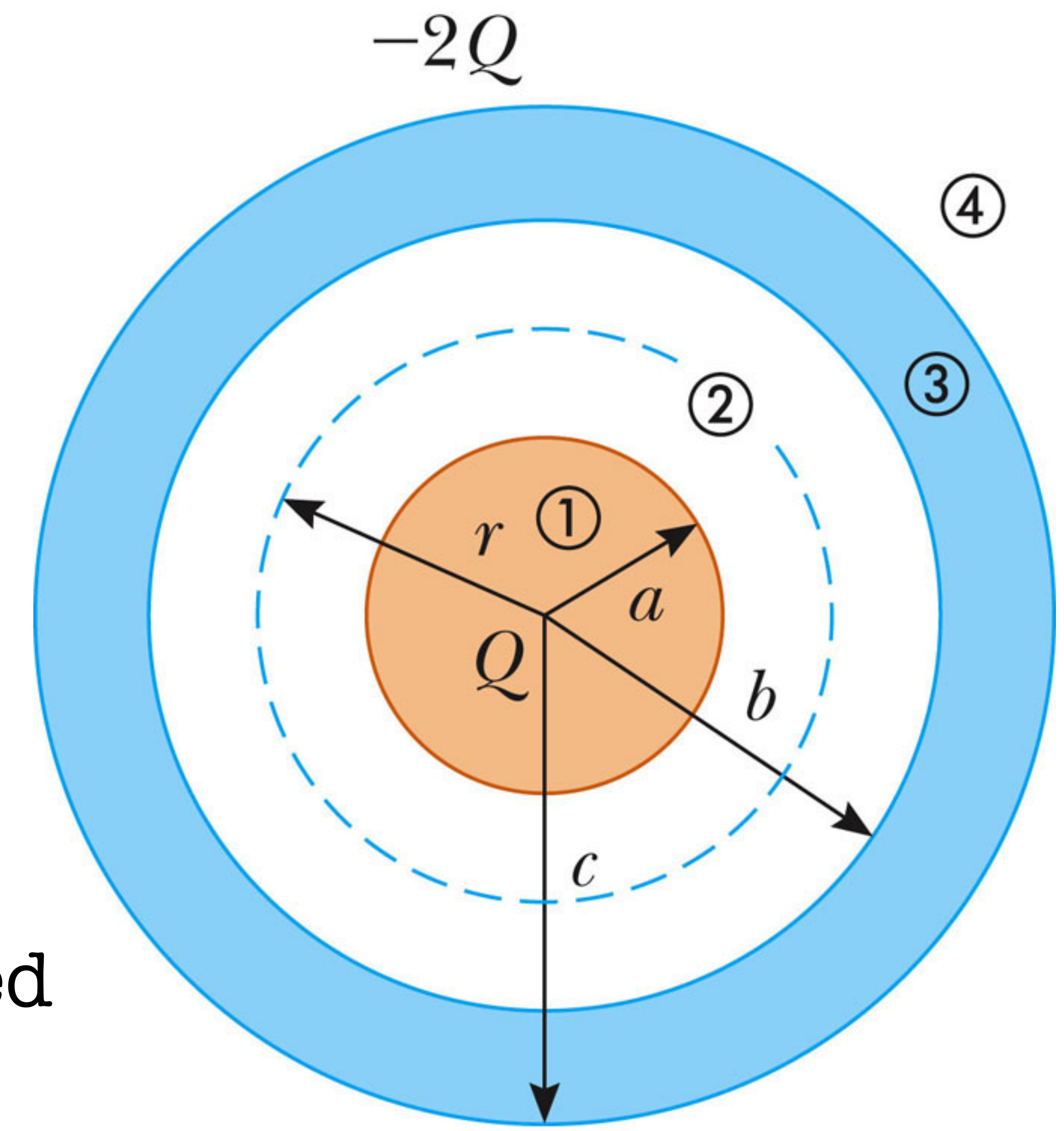
➢ Charge and dimensions as marked

Analyze

◉ System has spherical symmetry, Gauss Law problem type I

◉ Electric field inside conductors is zero

◉ There are two other ranges, $a < r < b$ and $b < r$ that need to be considered



➢ Arguments for electric field

◉ Similar to sphere example, because spherical symmetry, electrical field in these two ranges $a < r < b$ and $b < r$ is only a function of r, and goes along radius

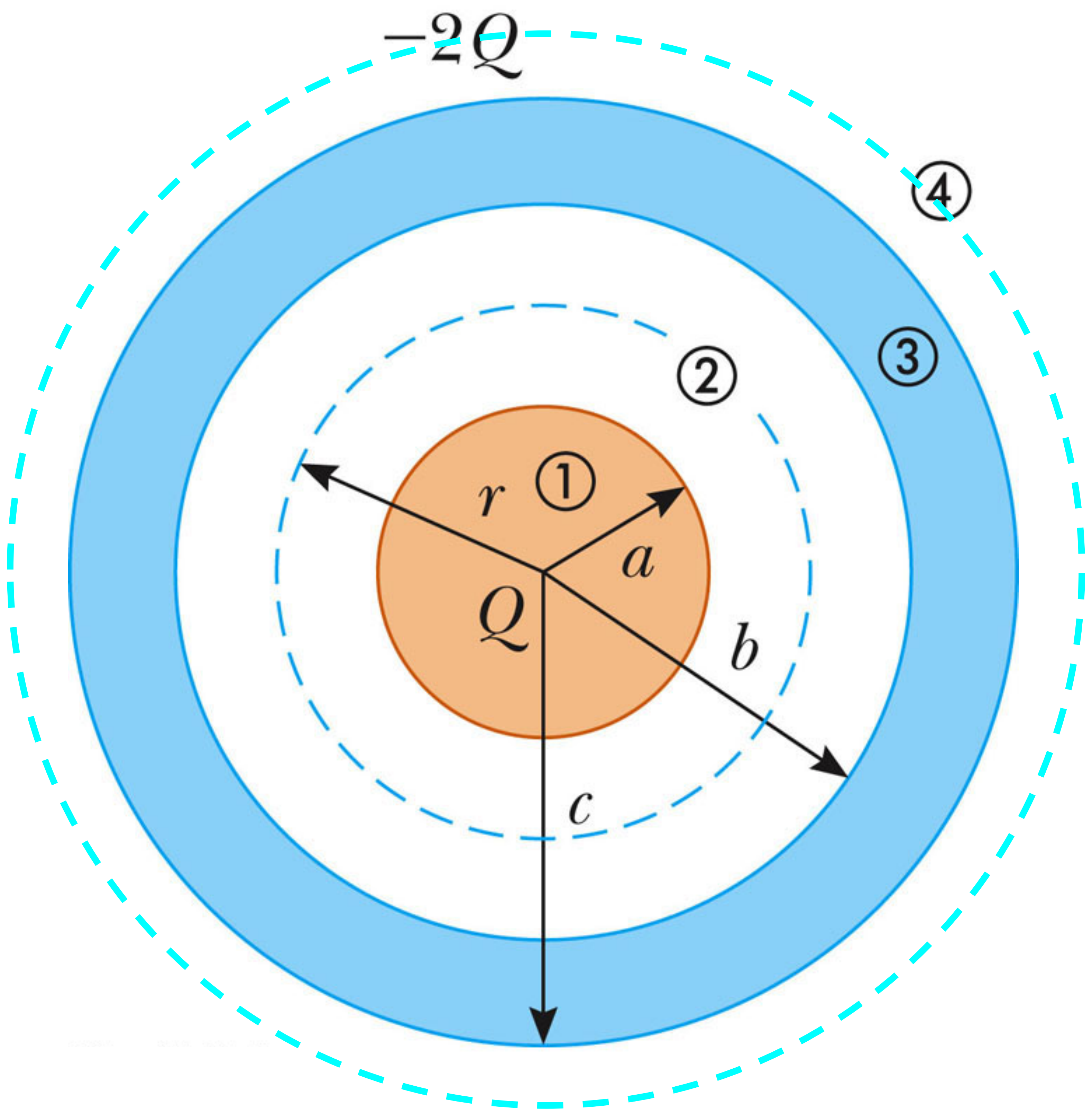
Construct Gaussian Surface & Calculate Flux & Use Gauss Law To Get Electric Field

- $E = 0$ when $r < a$, and $b < r < c$
- Construct a Gaussian sphere with its center coincides with center of inner sphere
- When $a < r < b$
  - Flux $\Phi_E = E \cdot 4\pi r^2$
  - Apply Gauss Law $\Phi_E = \dfrac{Q}{\epsilon_0}$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \quad \text{or} \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

- When $b < r$
  - Flux $\Phi_E = E \cdot 4\pi r^2$
  - Apply Gauss Law $\Phi_E = \dfrac{-2Q + Q}{\epsilon_0}$

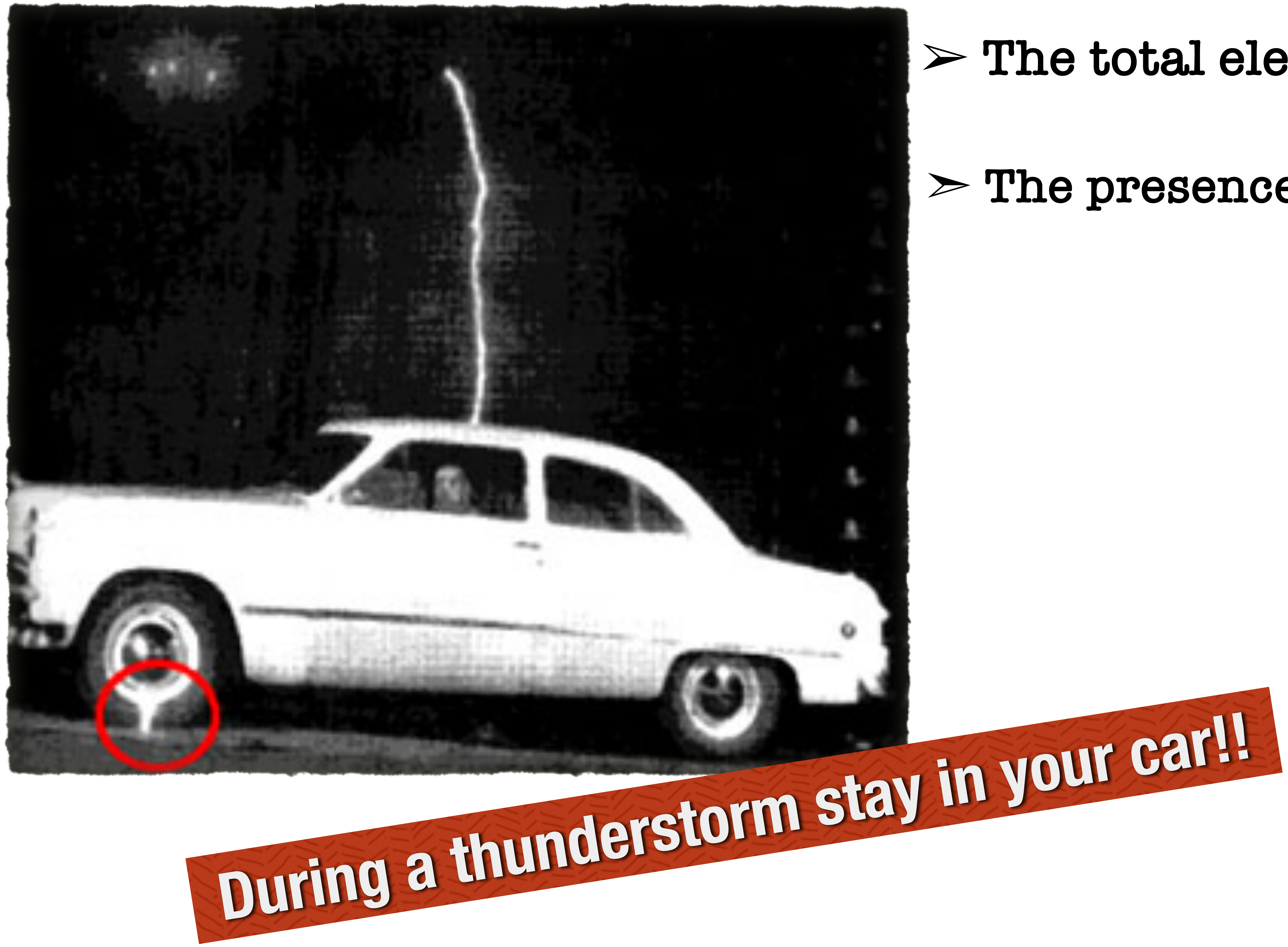
$$E = \frac{1}{4\pi\epsilon_0}\frac{-Q}{r^2} \quad \text{or} \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{-Q}{r^2}\hat{r}$$



Definition of a Faraday Cage

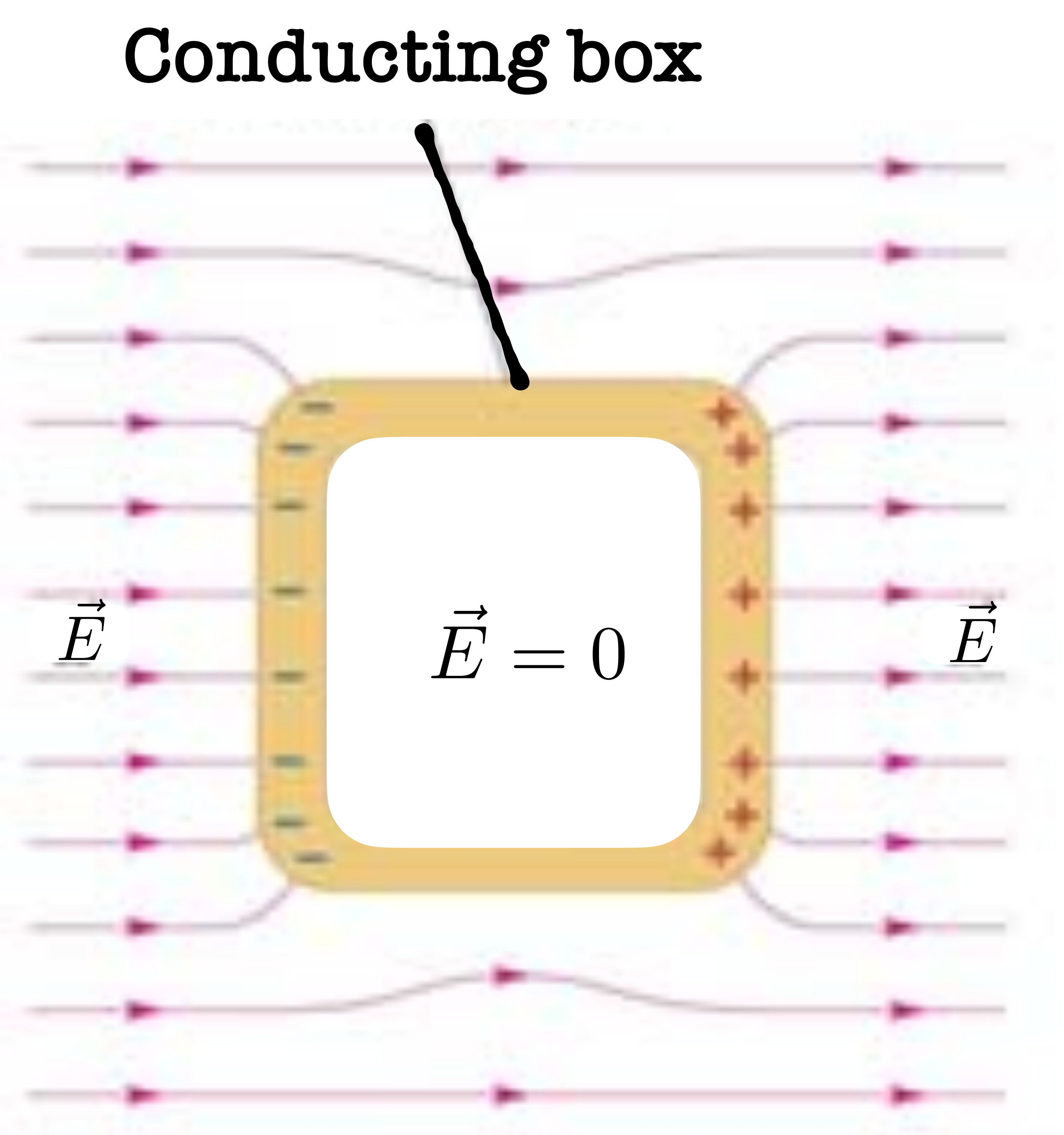
A Practical conclusión from Gauss's Law
Faraday's Cage

➢ The field induces charges on the left and right sides of the conductiva box

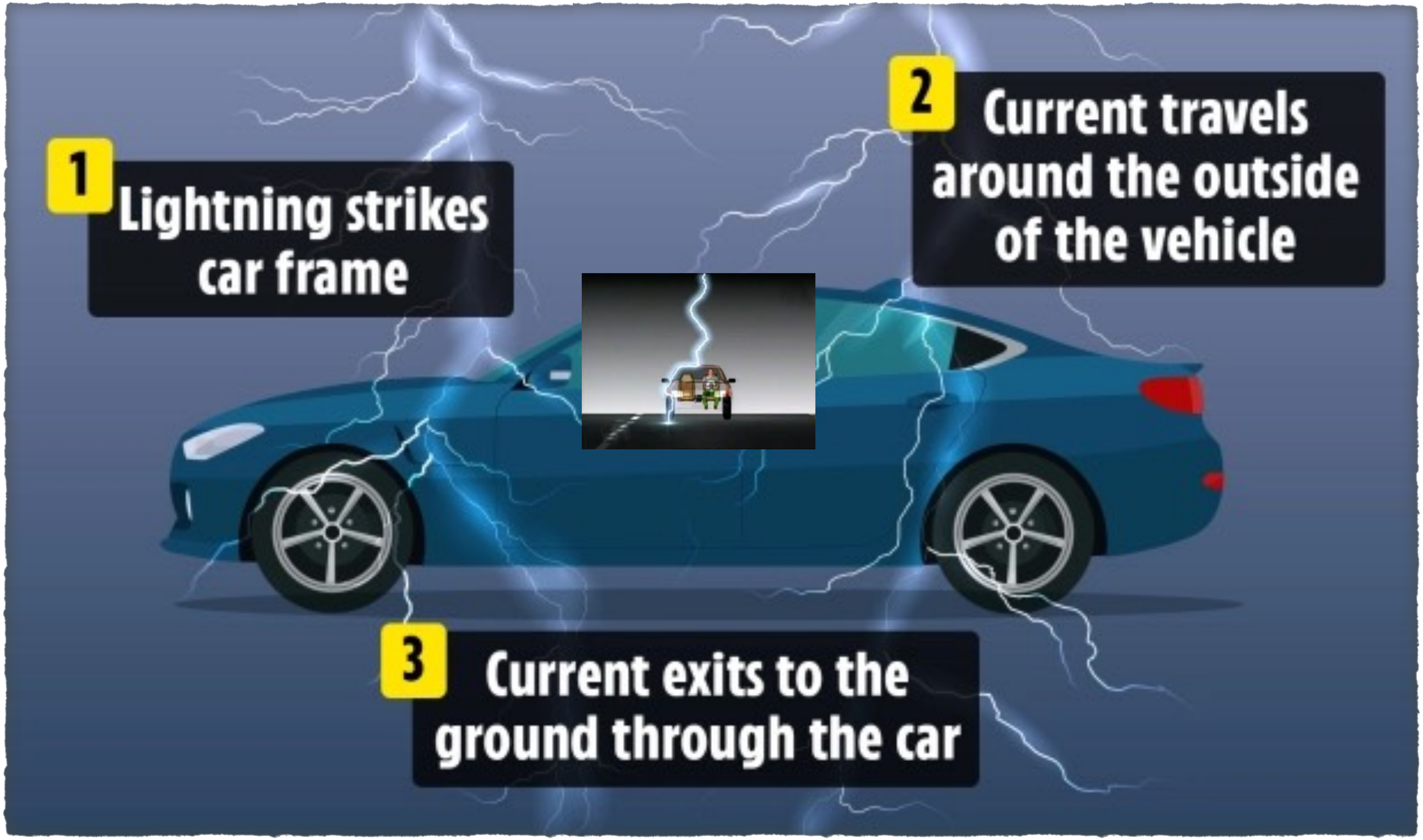


➢ The total electric field inside the box is zero

➢ The presence of the box distorts the Field in adjacent regions



When a Car is Struck by Lightning

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