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Problems set # 4 **Physics 167** **Solutions**

1. Eight equal charges q are located at the corners of a cube of side s. Find the electric potential at one corner, taking zero potential to be infinitely far away.

Solution: To compute the potential all you need to know is that there are 3 charges a distance s away, 3 a distance $s\sqrt{2}$ away, and one charge a distance $s\sqrt{3}$ away. You find the potential due to each charge separately, and add the results via superposition: $V = \frac{q}{4\pi\epsilon_0 s}\left(3 + \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right) \approx 5.69\frac{q}{4\pi\epsilon_0 s}$.

2. Four point charges are fixed at the corners of a square centered at the origin, as shown in Fig. 1. The length of each side of the square is $2a$. The charges are located as follows: $+q$ is at $(-a,+a)$, $+2q$ is at $(+a,+a)$, $-3q$ is at $(+a,-a)$, and $+6q$ is at $(-a,-a)$. A fifth particle that has a mass m and a charge $+q$ is placed at the origin and released from rest. Find its speed when it is a very far from the origin.

Solution: The diagram shows the four point charges fixed at the corners of the square and the fifth charged particle that is released from rest at the origin. We can use conservation of energy to relate the initial potential energy of the particle to its kinetic energy when it is at a great distance from the origin and the electrostatic potential at the origin to express $U_{\rm i}$. Use conservation of energy to relate the initial potential energy of the particle to its kinetic energy whenit is at a great distance from the origin: $\Delta K + \Delta U = 0$, or because $K_{\rm i} = U_{\rm f} = 0$, $K_{\rm f} - U_{\rm i} = 0$. Express the initial potential energy of the particle to its charge and the electrostatic potential at the origin: $U_{\rm i} = qV(0)$. Substitute for $K_{\rm f}$ and $U_{\rm i}$ to obtain: $\frac{1}{2}mv^2 - qV(0) = 0 \Rightarrow v = \sqrt{2qV(0)/m}$. Express the electrostatic potential at the origin: $V(0) = \frac{q}{4\pi\epsilon_0\sqrt{2}a}(1+2-3+6) = \frac{6q}{4\pi\epsilon_0\sqrt{2}a}$. Substitute for $V(0)$ and simplify to obtain: $v = q\sqrt{\frac{6\sqrt{2}}{4\pi\epsilon_0 ma}}$.

3. Five identical point charges $+q$ are arranged in two different manners as shown in Fig. 2: in once case as a face-centered square, in the other as a regular pentagon. Find the potential energy of each system of charges, taking the zero of potential energy to be infinitely far away. Express your answer in terms of a constant times the energy of two charges $+q$ separated by a distance a.

Solution: Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just: $PE_{\rm pair} = \frac{q^2}{4\pi\epsilon_0 d}$. Since all of the charges are the same in both configurations, all we need to do is figure out how many pairs there are in each situation, and for each pair, how far apart the charges are. How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force - which we will do as a check - but we can also calculate how many there are. In both configurations, we have 5 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair $(2, 1)$ is the same as $(1, 2)$. Pairings like this are known as combinations, as opposed to permutations where $(1, 2)$ and $(2, 1)$ are not the same. It is straightforward to see that the ways of choosing pairs from five charges $= \frac{5!}{2!(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = 10$. So there are 10 unique ways to choose 2 charges out of 5. First, let's consider the face-centered square configuration. In order to enumerate the possible pairings, we should label the charges to keep them straight. Label the corner charges $1-4$, and the center charge 5 (it doesn't matter which way you number the corners, just so long as 5 is the middle charge). Then our possible pairings are: $(1, 2)$ $(1, 3)$ $(1, 4)$ $(1, 5)$ $(2, 3)$ $(2, 4)$ $(2, 5)$ $(3, 4)$ $(3, 5)$ $(4, 5)$. There are ten, just as we expect. In this configuration, there are only three different distances that can separate a pair of charges: pairs on adjacent corners are a distance $a\sqrt{2}$ apart, a center-corner pairing is a distance a apart, and a far corner-far corner pair is $2a$ apart. We can take our list above, and sort it into pairs that have the same separation. We have four pairs of charges a distance a apart, four that are $a\sqrt{2}$ apart, and two that are $2a$ apart. Write down the energy for each type of pair listed in Table 1, multiply by the number of those pairs, and add the results together: $PE_{\text{square}} = 4(\text{center} - \text{corner pair}) + 2(\text{far corner pair}) + 4(\text{adjacent corner pair}) = \frac{q^2}{4\pi\epsilon_0 a}\left(4 + 1 + 4/\sqrt{2}\right) \approx 7.83\frac{q^2}{4\pi\epsilon_0 a}$. For the pentagon configuration, things are even easier. This time, just pick one charge as "1", and label the others from 2-5 in a clockwise or counter-clockwise fashion. Since we still have 5 charges, there are still 10 pairings, and they are the same as the list above. For the pentagon, however, there are only two distinct distances - either charges can be adjacent, and thus a distance a apart, or they can be next-nearest neighbors. What is the next-nearest neighbor distance? In a regular pentagon, each of the angles is $108°$, and in our case, each of the sides has length a, as shown in Fig. 2. We can use the law of cosines to find the distance d between next-nearest neighbors; $d^2 = a^2 + a^2 - 2a^2\cos 108° = 2a^2(1 - \cos 10°) \Rightarrow d = a\sqrt{2 - 2\cos 108°} = a\phi \approx 1.618a$, where the number ϕ is known as the "Golden Ratio." The distances a and d automatically satisfy the golden ratio in a regular pentagon, $d/a = \phi$. Given the nearest neighbor distance in terms of a, we can then create a table of pairings for the pentagon; these are listed in Table 2. Now once again we write down the energy for each type of pair, and multiply by the number of pairs: $PE_{\text{pentagon}} = 5(\text{energy of adjacent pair}) + 5(\text{energy of next} - \text{nearest neighbor pair}) = \frac{5q^2}{4\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{d}\right) = \frac{5q^2}{4\pi\epsilon_0 a}\left[1 + \frac{1}{\sqrt{2(1-\cos 180°)}}\right] \approx 8.09\frac{q^2}{4\pi\epsilon_0 a}$. So the energy of the pentagonal configuration is higher, meaning it is less favorable than the square configuration. Neither one is energetically favored though - since the energy is positive, it means that either configuration of charges is less stable than just having all five charges infinitely far from each other. This makes sense - if all five charges have the same sign, they don't want to arrange next to one another, and thus these arrangements cost energy to keep together. If we didn't force the charges together in these patterns, the positive energy tells us that they would fly apart given half a chance. For this reason, neither one is a valid sort of crystal lattice, real crystals have equal numbers of positive and negative charges, and are overall electrically neutral.

4. Consider a system of two charges shown in Fig. 3. Find the electric potential at an arbitrary point on the x axis and make a plot of the electric potential as a function of x/a.

<u>Solution</u> The electric potential can be found by the superposition principle. At a point on the x axis, we have $V(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{x-a} + \frac{1}{4\pi\epsilon_0}\frac{(-q)}{|x+a|} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right]$. The above expression may be

rewritten as $\frac{V(x)}{V_0} = \frac{1}{|x/a-1|} - \frac{1}{|x/a+1|}$, where $V_0 = \frac{q}{4\pi\epsilon_0 a}$. The plot of the dimensionless electric potential as a function of x/a is depicted in Fig. 3.

5. A point particle that has a charge of +11.1 nC is at the origin. *(i)* What is (are) the shapes of the equipotential surfaces in the region around this charge? *(ii)* Assuming the potential to be zero at $r = \infty$, calculate the radii of the five surfaces that have potentials equal to 20.0 V, 40.0 V, 60.0 V, 80.0 V and 100.0 V, and sketch them to scale centered on the charge. *(iii)* Are these surfaces equally spaced? Explain your answer. *(iv)* Estimate the electric field strength between the 40.0-V and 60.0-V equipotential surfaces by dividing the difference between the two potentials by the difference between the two radii. Compare this estimate to the exact value at the location midway between these two surfaces.

Solution: *(i)* The equipotential surfaces are spheres centered on the charge. *(ii)* The electric potential difference due to a point charge is given by $V_b - V_a = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$. Taking the potential to be zero at $r_a = \infty$ yields: $V_b - 0 = \frac{1}{4\pi\epsilon_0}\frac{1}{r_b} \Rightarrow V = \frac{Q}{4\pi\epsilon_0 r} \Rightarrow r = \frac{Q}{4\pi\epsilon_0 V}$. Because $Q = 1.11 \times 10^{-8}$ C, it follows that $r = 8.988 \times 10^9 \ \mathrm{N \cdot m^2/C^2} \ 1.11 \times 10^{-8} \ \mathrm{C}\frac{1}{V}$. Now you can use the previous equation to determine the values of r:

V [V]	20.0	40.0	60.0	80.0	100.0
r [m]	4.99	2.49	1.66	1.25	1.00

The equipotential surfaces are shown in cross-section in Fig. 4. *(iii)* No. The equipotential surfaces are closest together where the electric field strength is greatest. *(iv)* The average value of the magnitude of the electric field between the 40.0-V and 60.0-V equipotential surfaces is given by: $E = -\frac{\Delta V}{\Delta r} = -\frac{40 \ \mathrm{V} - 60 \ \mathrm{V}}{2.49 \ \mathrm{m} - 1.66 \ \mathrm{m}} \simeq 29\frac{\mathrm{V}}{\mathrm{m}}$. The exact value of the electric field at the location midway between these two surfaces is given by $E = \frac{Q}{4\pi\epsilon_0 r^2}$, where r is the average of the radii of the 40.0-V and 60.0-V equipotential surfaces. Substitute numerical values and evaluate $E_{\rm exact} = \frac{8.988 \times 10^9 \ \mathrm{N \cdot m^2/C^2} \ 1.11 \times 10^{-8} \ \mathrm{C}}{(1.66 \ \mathrm{m} + 2.49 \ \mathrm{m})^2/4} \simeq 23\frac{\mathrm{V}}{\mathrm{m}}$. The estimated value for E differs by about 21% from the exact value.

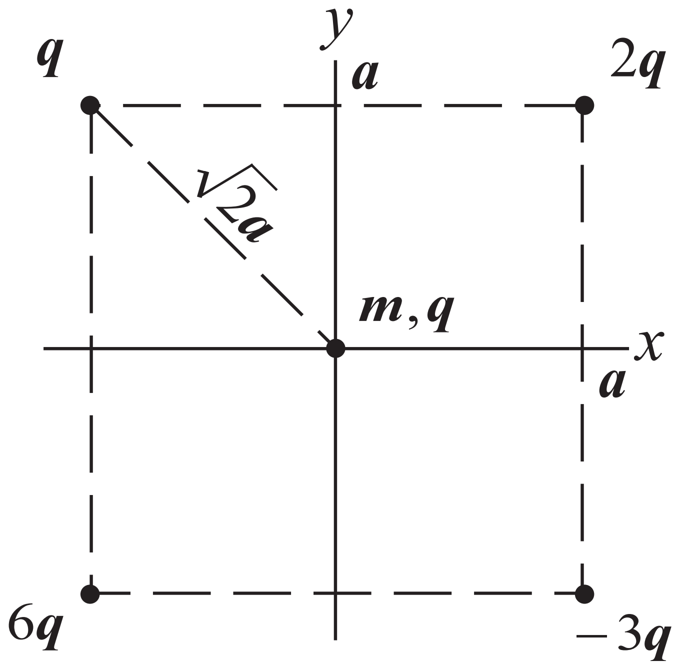

Figure 1: Problem 2.

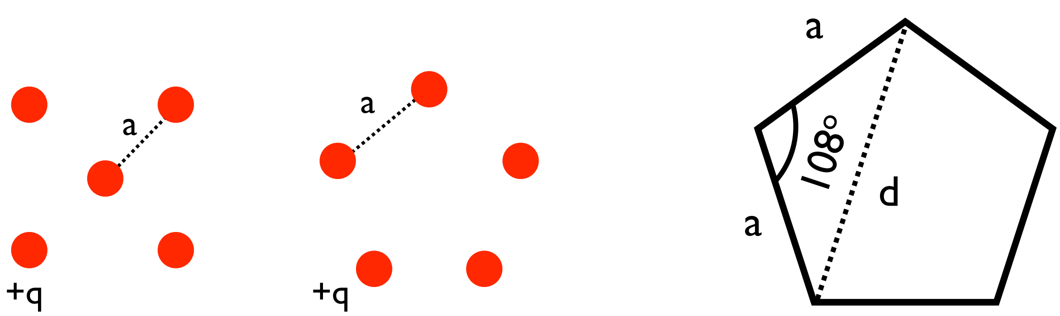

Figure 2: Problem 3.

Table 1: *Charge pairings in the square lattice*

#, pairing type	**separation**				**pairs**
4, center-corner	a	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$
4, adjacent corners	$a\sqrt{2}$	$(1,4)$	$(3,4)$	$(2,3)$	$(1,2)$
2, far corner	$2a$			$(1,3)$	$(2,4)$

Table 2: *Charge pairings in the pentagonal lattice*

#, pairing type	separation					pairs
5, next-nearest neighbors	d	$(1,3)$	$(1,4)$	$(2,4)$	$(2,5)$	$(3,5)$
5, adjacent	a	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,1)$

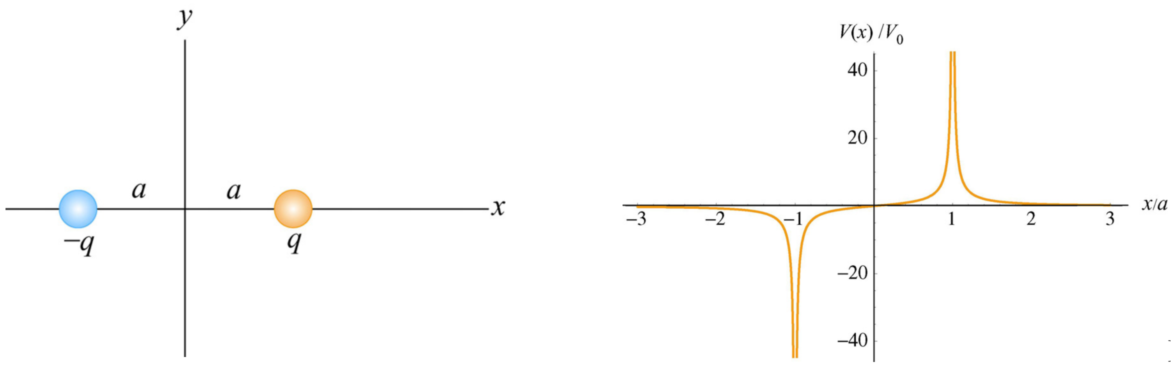

Figure 3: The lectric dipole of problem 4.

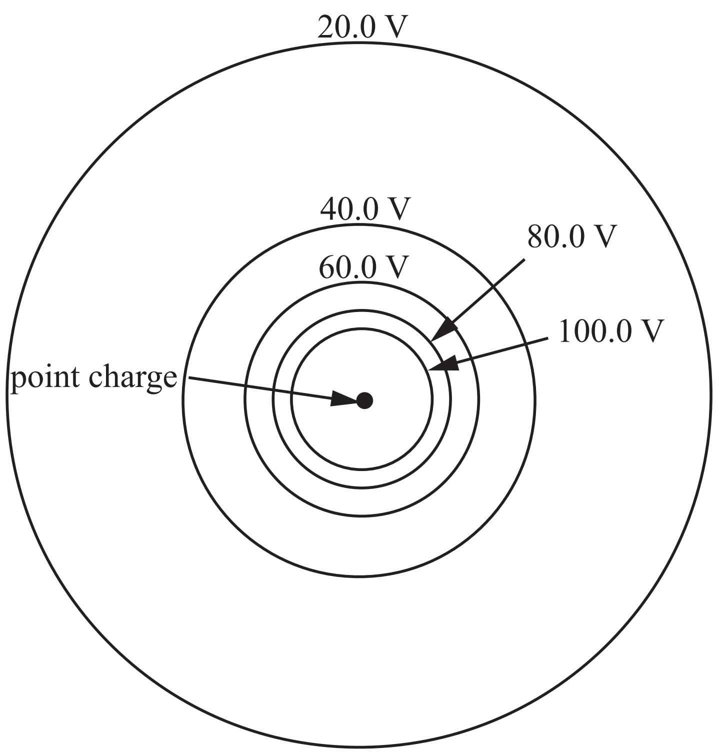

Figure 4: Problem 5.